

Lévy Laplacian for Square Roots of Measures

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Abstract

L. Accardi showed that a Banach space of signed measures is homeomorphic to a Hilbert space formed by the so-called square roots of measures. In this paper, we redefine square roots of measures in view of the theory of measures on infinite dimensional spaces, and introduce notions of differentiation, Fourier transform, and convolution product of square roots of measures, and examine those relations. By using these tools, we study the Lévy Laplacian for squares root of measures including non-Gaussian type. It is shown that the symbol of the Lévy Laplacian is equal to the quadratic variation of paths.

1 Introduction

L. Accardi shows that a Banach space of signed measures is homeomorphic to a Hilbert space formed by the so-called square roots of measures [1]. The notion of the square roots of measures is introduced to interpret the result of Segal [21] and Nelson [16] in a context more general than that of Gaussian measures. In this paper we first discuss about square roots of measures on \mathbb{R}^∞ : the countable product space of real lines, and we redefine the square roots of measures in view of the structure of the projective limit. The advantage of adopting the new definition is to make available a Fourier transform for the square roots of measures. It is defined as a translation of the concept of the adjoint measure studied by Yamasaki [28]. For a more detailed discussion about the relationship between the ordinary definition and our definition of square roots of measures, see [10].

A typical square root of a measure is a sequence of square roots of density functions of finite dimensional projection of a positive bounded Borel measure μ on \mathbb{R}^∞ . We will denote it by $\sqrt{\mu}$ and call it a square root of μ . With this notation it follows easily that the inner product of $\sqrt{\mu}$ and $\sqrt{\nu}$ coincide with the Hellinger integral

$$H(\mu, \nu) = \int_{\mathbb{R}^\infty} \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda$$

where λ stands for any positive bounded Borel measure on \mathbb{R}^∞ with respect to which both μ and ν are absolutely continuous.

We also introduce a notion of the directional differentiation for square roots of measures with the aid of the theory of differentiable measures studied by Averbuh-Smolyanov-Fomin [2], Skorohod [22], and many other authors. The differentiation for square roots of measures inherit the property of differentiation from ordinary differentiable measures. In suitable conditions, the Fourier transform is compatible with the differentiation defined here in the following sense: if f is a differentiable square root of a measure in the direction $\rho \in \mathbb{R}^\infty$, then the Fourier transform of its directional derivative is given by $2\pi\sqrt{-1} \langle \xi, \rho \rangle \times \hat{f}$. Here $\langle \xi, \rho \rangle = \sum_{k=1}^{\infty} \xi_k \rho_k$ and \hat{f} stands for the Fourier transform of f .

The main purpose of the present paper is to study the Lévy Laplacian for square roots of measures on $C_0[0, T]$: the space of real-valued continuous functions on $[0, T]$ which is 0 at the origin. The Lévy Laplacian is defined as the Cèsaro mean of second order differential operators:

$$\Delta_L = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\partial^2}{\partial x_n^2},$$

where x_1, x_2, \dots constitute a coordinate system of the infinite dimensional vector space under consideration. There are many results about the Lévy Laplacian associated with the Gaussian measures and Brownian functionals ([8], [14], [17]). But in this paper we will be concerned with the measures which are not necessarily of Gaussian type and give a sufficient condition for square roots of measures on which the Lévy Laplacian acts naturally. The key is the Fourier transform for square roots of measures. To transform the Lévy Laplacian into a multiplication operator by a function makes it easy to examine the domain of the Lévy Laplacian. In fact, it is shown that the function is equal to the quadratic variation of paths on $[0, T]$. This result will be useful for the theory of Sobolev spaces and pseudo-differential operators of square roots of measures. However these topics exceed the scope of this paper.

2 Definition and Properties of Square Roots of Measures

Let \mathbb{R}^∞ be countable direct product of real lines and let d be the distance of \mathbb{R}^∞ defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \quad (x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty} \in \mathbb{R}^\infty). \quad (2.1)$$

Then (\mathbb{R}^∞, d) is a complete separable metric space. Since \mathbb{R} is a nuclear space, so is \mathbb{R}^∞ . (see [4], [20], and [28] for more details).

A sequence of positive L^2 -functions $\{f_n\}$, $f_n \in L^2(R^n)$ is called a superprojective system of L^2 -functions if

$$\int_R |f_{n+1}(x, x')|^2 dx' \leq |f_n(x)|^2 \quad \text{a.e. } x \in R^n (n = 1, 2, \dots). \quad (2.2)$$

The condition (2.2) is called weakly L^2 -compatibility condition. If the inequality (2.2) is replaced with an equality, $\{f_n\}$ is called a projective system of L^2 -functions and (2.2) is called L^2 -compatibility condition. We will denote by $\mathcal{L}_b^2(\mathbb{R}^\infty)$ the totality of superprojective systems of

L^2 -functions. For $\{f_n\}, \{g_n\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ and $\alpha \in C$ addition and scalar multiplication is defined by

$$\{f_n\} + \{g_n\} = \{f_n + g_n\}, \alpha\{f_n\} = \{\alpha f_n\}$$

respectively. Then we will denote by $\mathcal{L}^2(\mathbb{R}^\infty)$ the complex linear hull of $\mathcal{L}_b^2(\mathbb{R}^\infty)$. $\mathcal{L}_b^2(\mathbb{R}^\infty)$ is essentially expressed as the sum of four superprojective systems.

Proposition 2.1. *For all $f \in \mathcal{L}^2(\mathbb{R}^\infty)$ there exist $f^j \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($j = 1, 2, 3, 4$) such that*

$$f = f^1 - f^2 + \sqrt{-1}(f^3 - f^4). \quad (2.3)$$

Proof. Let $f \in \mathcal{L}^2(\mathbb{R}^\infty)$. By definition, there are $\alpha_k \in C$ and $f^k \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($1 \leq k \leq n$) such that $f = \sum_{k=1}^n \alpha_k f^k$. For $\alpha = a + b\sqrt{-1}$, $a, b \in R$ and $f \in \mathcal{L}_b^2(\mathbb{R}^\infty)$, we have

$$\alpha f = (a \vee 0)f - (-a \vee 0)f + \sqrt{-1}(b \vee 0)f - \sqrt{-1}(-b \vee 0)f,$$

where $(a \vee 0) = \max\{a, 0\}$. This shows αf is of the form such as (2.3). Thus there are $g^{(j,k)} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($j = 1, 2, 3, 4$, $1 \leq k \leq n$) such that $\alpha_k f^k = g^{(1,k)} - g^{(2,k)} + \sqrt{-1}(g^{(3,k)} - g^{(4,k)})$ and f is expressed as

$$\sum_{k=1}^n g^{(1,k)} - \sum_{k=1}^n g^{(2,k)} + \sqrt{-1} \left(\sum_{k=1}^n g^{(3,k)} - \sum_{k=1}^n g^{(4,k)} \right).$$

Here $\sum_{k=1}^n g^{(j,k)} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($j = 1, 2, 3, 4$). In fact, for $f^1 = \{f_n^1\}$ and $f^2 = \{f_n^2\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$, we have

$$\begin{aligned} & \int_R |f_{n+1}^1(x, x') + f_{n+1}^2(x, x')|^2 dx' \\ &= \int_R (|f_{n+1}^1(x, x')|^2 + |f_{n+1}^2(x, x')|^2 + 2f_{n+1}^1(x, x')f_{n+1}^2(x, x')) dx', \\ &\leq \int_R (|f_{n+1}^1(x, x')|^2 + |f_{n+1}^2(x, x')|^2) dx' + 2\sqrt{\int_R |f_{n+1}^1(x, x')|^2 dx'} \sqrt{\int_R |f_{n+1}^2(x, x')|^2 dx'} \\ &\leq (f_n^1(x))^2 + (f_n^2(x))^2 + 2f_n^1(x)f_n^2(x) = |f_n^1(x) + f_n^2(x)|^2. \end{aligned}$$

This shows that $f^1 + f^2 \in \mathcal{L}_b^2(\mathbb{R}^\infty)$. Thus the finite sum of superprojective systems is also a superprojective system. \square

For a topological space X we will denote by $\mathcal{B}(X)$ the totality of Borel sets of X . Let $f = \{f_n\}, g = \{g_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ and $n \geq 1$. We consider a sequence of L^1 -functions $\{h_k^n\}$ defined by

$$h_k^n(x) = \int_{R^k} f_{n+k}(x, x') g_{n+k}(x, x') dx' \quad (k = 1, 2, \dots). \quad (2.4)$$

We first examine $\{h_k^n\}$ if $f, g \in \mathcal{L}_b^2(\mathbb{R}^\infty)$. In this case, (2.4) is a sequence of non-negative L^1 -functions such that $h_{k+1}^n(x) \leq h_k^n(x)$ ($k = 1, 2, \dots$). Hence by the monotone convergence theorem, there exists $h^n \in L^1(\mathbb{R}^n)$, $h_n \geq 0$ such that $\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |h_k^n(x) - h^n(x)| dx = 0$.

In addition $\{h_n\}$ satisfies

$$\int_{\mathbb{R}^n} h_{n+1}(x, x') dx' = h_n(x) \quad (n = 1, 2, \dots). \quad (2.5)$$

The Kolmogorov extension theorem ensures that there exists a bounded positive Borel measure μ on \mathbb{R}^∞ such that

$$\mu(p_n^{-1}(E)) = \int_E h_n(x) dx \quad (2.6)$$

for all $E \in \mathcal{B}(\mathbb{R}^n)$. Here p_n stand for the projection from \mathbb{R}^∞ to \mathbb{R}^n defined by

$$p_n : \mathbb{R}^\infty \ni (x_1, x_2, \dots) \mapsto (x_1, x_2, \dots, x_n).$$

When $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$, $\{h_n^k\}$ converges to some $h_n \in L^1(\mathbb{R}^n)$ because it is a linear combination of monotone decreasing sequences of L^1 -functions. Since $\{h_n\}$ is a complex linear combination of the sequences like (2.5), there also exists a complex Borel measure μ satisfying (2.6).

The measure μ defined above is called the product of f and g ; it is denoted by $f \cdot g$. In particular we write $|f|^2 = f \cdot \bar{f}$ as the square of f . Here $\bar{f} = \{\bar{f}_n\}$ denotes the complex conjugate of f . In this sense the square of $\mathcal{L}^2(\mathbb{R}^\infty)$ is regarded as a measure and the element of $\mathcal{L}^2(\mathbb{R}^\infty)$ a square root of a measure on \mathbb{R}^∞ .

Definition 2.2. Assume that μ is a positive Borel measure on \mathbb{R}^∞ such that the finite dimensional projections $p_n(\mu) = \mu(p_n^{-1}(E))$, $E \in \mathcal{B}(\mathbb{R}^n)$ is absolutely continuous with the Lebesgue measure and let f_n be a density function of $p_n(\mu)$. We call $\{\sqrt{f_n}\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ as the square root of μ and denote it by $\sqrt{\mu}$.

Proposition 2.3. Fix $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$ and $\alpha \in \mathbb{C}$. Then the following (1) – (3) hold.

- (1) $f \cdot g = g \cdot f$
- (2) $(f + g) \cdot h = f \cdot h + g \cdot h$
- (3) $(\alpha f) \cdot g = \alpha(f \cdot g)$

Proposition 2.4. Let $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$.

$$|f \cdot g|(E) \leq \sqrt{|f|^2(E)} \sqrt{|g|^2(E)} \quad (2.7)$$

holds for any $E \in \mathcal{B}(\mathbb{R}^\infty)$. Here $|f \cdot g|$ denotes the total variation of complex Borel measures $f \cdot g$.

Proof. Let us first prove the case E is of the form $p_n^{-1}(E_n)$, $E_n \in \mathcal{B}(\mathbb{R}^n)$. Let h_n^k and h_n are the function defined by (2.4).

$$\begin{aligned} |f \cdot g(p_n^{-1}(E_n))| &\leq \limsup_{k \rightarrow \infty} \int_{E_n} \int_{\mathbb{R}^k} |f_{n+k}(x, x') g_{n+k}(x, x')| dx' dx \\ &\leq \lim_{k \rightarrow \infty} \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |f_{n+k}(x, x')|^2 dx' dx} \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}(x, x')|^2 dx' dx} \\ &= \sqrt{|f|^2(p_n^{-1}(E_n))} \sqrt{|g|^2(p_n^{-1}(E_n))}, \end{aligned}$$

which proves (2.7).

If K is a compact set of \mathbb{R}^∞ ,

$$K = \bigcap_{n=1}^{\infty} p_n^{-1}(p_n(K))$$

holds (see for instance [4]). Thus we have

$$\begin{aligned} f \cdot g(K) &= f \cdot g \left(\bigcap_{n=1}^{\infty} p_n^{-1}(p_n(K)) \right) = \lim_{k \rightarrow \infty} f \cdot g(p_k^{-1}(p_k(K))) \\ &\leq \lim_{k \rightarrow \infty} \sqrt{|f|^2(p_k^{-1}(p_k(K)))} \sqrt{|g|^2(p_k^{-1}(p_k(K)))} = \sqrt{|f|^2(K)} \sqrt{|g|^2(K)}. \end{aligned}$$

Since a complex Borel measure on a separable complete metric space is tight (see for instance [7], [18]) and a countable product of separable spaces is also separable, complex Borel measures on \mathbb{R}^∞ are tight. So for all $\epsilon > 0$ and $E \in \mathcal{B}(\mathbb{R}^\infty)$ there exists a compact set $K \subset E$ of \mathbb{R}^∞ such that

$$|f \cdot g|(E) - |f \cdot g|(K) \leq |f \cdot g|(E \setminus K) \leq \epsilon.$$

It shows that (2.7) holds for $E \in \mathcal{B}(\mathbb{R}^\infty)$.

□

Set $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$. A sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}^2} : \mathcal{L}^2(\mathbb{R}^\infty) \times \mathcal{L}^2(\mathbb{R}^\infty) \mapsto \mathbb{C}$ is defined by $\langle f, g \rangle_{\mathcal{L}^2} = (f \cdot \bar{g})(\mathbb{R}^\infty)$ and a seminorm is defined by $\|f\|_{\mathcal{L}^2} = \sqrt{\langle f, f \rangle_{\mathcal{L}^2}}$. f is said to be equivalent to g if $\|f - g\|_{\mathcal{L}^2} = 0$. Inequality (2.7) shows that the equivalence relation is well defined. Let $L^2(\mathbb{R}^\infty)$ be the quotient set of $\mathcal{L}^2(\mathbb{R}^\infty)$ by this equivalence relation. $L^2(\mathbb{R}^\infty)$ is a pre-Hilbert space by the inner product $\langle \cdot, \cdot \rangle_{L^2}$ induced by $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$. Here we write $\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}}$. We will show that $L^2(\mathbb{R}^\infty)$ is a Hilbert space. The following three lemmas are needed to prove.

Lemma 2.5. *For all $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$, $f_n(x) \geq 0$, there exists $g = \{g_n\}$ which is equivalent to f and satisfy the L^2 -compatibility condition i.e.*

$$\int_{\mathbb{R}} |g_{n+1}(x, x')|^2 dx' = |g_n(x)|^2 \quad \text{a.e. } x \in \mathbb{R}^n.$$

Proof. Assume that f is of the form $f = f^1 - f^2, f^1 = \{f_n^1\}, f^2 = \{f_n^2\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$, $f_n^1 - f_n^2 \geq 0$. Set $g_n(x) = \sqrt{\lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} |f_{n+k}(x, x')|^2 dx'}$. It is easy to check that $g = \{g_n\}$ satisfies L^2 -compatibility condition. $f \sim g$ is shown by

$$\begin{aligned} \|f - g\|_{\mathcal{L}^2}^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |f_n(x) - g_n(x)|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} ||f_n(x)|^2 - |g_n(x)|^2| dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \left| |f_n^1(x) - f_n^2(x)|^2 - \lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} |f_{n+k}^1(x, x') - f_{n+k}^2(x, x')|^2 dx' \right| dx \\ &\leq \sum_{j=1}^2 \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \left(|f_n^j(x)|^2 - \lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} |f_{n+k}^j(x, x')|^2 dx' \right) dx \\ &\quad + 2 \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} \left(f_n^1(x) f_n^2(x) - \lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} f_{n+k}^1(x, x') f_{n+k}^2(x, x') dx' \right) dx = 0. \end{aligned}$$

Here we have used

$$|a - b|^2 \leq |a^2 - b^2| \quad (a, b \geq 0).$$

□

Lemma 2.6. *Let $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$. Then there exist $h^j = \{h_n^j\} \in \mathcal{L}^2(\mathbb{R}^\infty)$, $h_n^j(x) \geq 0$ ($j = 1, 2, 3, 4$) such that*

$$(1) \quad h = h^1 - h^2 + \sqrt{-1}(h^3 - h^4) \in \mathcal{L}^2(\mathbb{R}^\infty) \text{ is equivalent to } f,$$

$$(2) \quad h_n^1(x) h_n^2(x) = h_n^3(x) h_n^4(x) = 0, \text{ a.e. } x \in \mathbb{R}^n.$$

Proof. By Proposition 2.1, there exist $\{f_n^j\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($j = 1, 2, 3, 4$) such that $f = f^1 - f^2 + \sqrt{-1}(f^3 - f^4)$. And lemma 2.5 shows that there exist $g^j = \{g_n^j\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ ($j = 1, 2, 3, 4$) such that

$$g^j \sim f^j, \quad \int_{\mathbb{R}} |g_{n+1}^j(x, x')|^2 dx' = |g_n^j(x)|^2 \quad \text{a.e. } x \in \mathbb{R}^n \quad (j = 1, 2, 3, 4).$$

Now set

$$H_n(x) = g_n^1(x) + g_n^2(x) - |g_n^1(x) - g_n^2(x)|.$$

Due to

$$\begin{aligned} |H_n(x)|^2 &= |g_n^1(x) + g_n^2(x)|^2 + |g_n^1(x) - g_n^2(x)|^2 - 2||g_n^1(x)|^2 - |g_n^2(x)|^2| \\ &= 2(|g_n^1(x)|^2 + |g_n^2(x)|^2) - 2||g_n^1(x)|^2 - |g_n^2(x)|^2|, \end{aligned}$$

it follows that

$$\begin{aligned}
& \int_{\mathbb{R}} |H_{n+1}(x, x')|^2 dx' \\
&= 2 \int_{\mathbb{R}} (|g_{n+1}^1(x, x')|^2 + |g_{n+1}^2(x, x')|^2) dx' - 2 \int_{\mathbb{R}} (|g_{n+1}^1(x, x')|^2 - |g_{n+1}^2(x, x')|^2) dx' \\
&\leq 2(|g_n^1(x)|^2 + |g_n^2(x)|^2) - 2 \left| \int_{\mathbb{R}} (|g_{n+1}^1(x, x')|^2 - |g_{n+1}^2(x, x')|^2) dx' \right| \\
&= 2(|g_n^1(x)|^2 + |g_n^2(x)|^2) - 2|g_n^1(x)|^2 - |g_n^2(x)|^2 = |H_n(x)|^2.
\end{aligned}$$

This shows that $\{H_n\}$ is the superprojective system. Thus $\frac{|g_n^1 - g_n^2| \pm (g_n^1 - g_n^2)}{2}$ belongs to $\mathcal{L}^2(\mathbb{R}^\infty)$. In the same way, we can show that $\frac{|g_n^3 - g_n^4| \pm (g_n^3 - g_n^4)}{2}$ belongs to $\mathcal{L}^2(\mathbb{R}^\infty)$. Here by letting

$$\begin{aligned}
h^1 &= \frac{|g_n^1 - g_n^2| + (g_n^1 - g_n^2)}{2}, \quad h^2 = \frac{|g_n^1 - g_n^2| - (g_n^1 - g_n^2)}{2}, \\
h^3 &= \frac{|g_n^3 - g_n^4| + (g_n^3 - g_n^4)}{2}, \quad h^4 = \frac{|g_n^3 - g_n^4| - (g_n^3 - g_n^4)}{2},
\end{aligned}$$

we can show that both of (1) and (2) are satisfied. \square

For a topological space X , let $M(X)$ be the totality of complex Borel measures. $M(X)$ is a complete metric space with the total variation norm $\|\mu\| = \sup_{E \in \mathcal{B}(X)} |\mu(E)|$.

Lemma 2.7. *Let $M_0(\mathbb{R}^\infty)$ be a collection of complex Borel measures whose finite dimensional projections $p_n(\mu) = \mu(p_n^{-1}(E_n))$ ($E_n \in \mathcal{B}(\mathbb{R}^n)$) are absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^n . Then $M_0(\mathbb{R}^\infty)$ is a closed subspace of $M(\mathbb{R}^\infty)$.*

Proof. Let $\{\mu^j\}$ be a Cauchy sequence of $M_0(\mathbb{R}^\infty)$. Since the totality of complex Borel measures on metric spaces are complete with the total variation norm (see for instance [18]), μ^j converges to μ : a complex measure on \mathbb{R}^∞ . Let g_n^j be the density function of $p_n(\mu_j)$. Then

$$\int_{\mathbb{R}^n} |g_n^j(x) - g_n^l(x)| dx \leq \|\mu^j - \mu^l\|.$$

Therefore for all n , g_n^j is a Cauchy sequence of $L^1(\mathbb{R}^n)$. This shows that the density function of $p_n(\mu)$ is $\lim_{n \rightarrow \infty} g_n^j(x)$. \square

Theorem 2.8. *$L^2(\mathbb{R}^\infty)$ is a complete metric space with respect to $\|\cdot\|_{L^2}$.*

Proof. Let $f^l = \{f_n^l\}$ is a Cauchy sequence with respect to $\|\cdot\|_{L^2}$. Lemma 2.6 ensures that there exists the representative element of f^l which is of the form $h^{l,1} - h^{l,2} + i(h^{l,3} - h^{l,4})$ that $h^{l,1} = \{h_n^{l,j}\}, h_n^{l,j}(x) \geq 0$ ($j = 1, 2, 3, 4$) belong to $\mathcal{L}^2(\mathbb{R}^\infty)$ and $h_n^{l,1}(x)h_n^{l,2}(x) = h_n^{l,3}(x)h_n^{l,4}(x) = 0$ for each $l \geq 1$. For all $l, m \in \mathbb{N}$,

$$\begin{aligned} \|f^l - f^m\|_{L^2}^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |(h_n^{l,1} - h_n^{l,2}) - i(h_n^{l,3} - h_n^{l,4}) \\ &\quad - (h_n^{m,1} - h_n^{m,2}) + i(h_n^{m,3} - h_n^{m,4})|^2 dx \geq \sum_{j=1}^4 \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |h_n^{l,j} - h_n^{m,j}|^2 dx. \end{aligned}$$

From this it follows that $\{[h^{l,j}]\}$ ($j = 1, 2, 3, 4$) are Cauchy sequences with respect to $\|\cdot\|_{L^2}$. Here $[f]$ denote the equivalence class to which f belongs. Hence by Lemma 2.5, $\{[h^{l,j}]\}$ is equivalent to a projective system of L^2 -functions. Thus we can assume that f^l is an equivalence class to which a projective system $g^l = \{g_n^l\}$ belongs for all $l \in \mathbb{N}$ without loss of generality.

If E is of the form $p_n^{-1}(E_n)$ ($E_n \in \mathcal{B}(\mathbb{R}^n)$), we obtain

$$\begin{aligned} &| |g^l|^2(E) - |g^m|^2(E) | \\ &= \left| \lim_{k \rightarrow \infty} \left(\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^l(x, x')|^2 dx dx' - \int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^m(x, x')|^2 dx dx' \right) \right| \\ &= \lim_{k \rightarrow \infty} \left(\sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^l(x, x')|^2 dx dx'} + \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^m(x, x')|^2 dx dx'} \right) \\ &\quad \times \left| \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^l(x, x')|^2 dx dx'} - \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^m(x, x')|^2 dx dx'} \right| \\ &\leq (\|g^l\|_{L^2} + \|g^m\|_{L^2}) \lim_{k \rightarrow \infty} \sqrt{\int_{E_n} \int_{\mathbb{R}^k} |g_{n+k}^l(x, x') - g_{n+k}^m(x, x')|^2 dx dx'} \\ &= (\|g^l\|_{L^2} + \|g^m\|_{L^2}) \|g^l - g^m\|_{L^2} \end{aligned}$$

by using the triangle inequality $|\|f\|_{L^2} - \|g\|_{L^2}| \leq \|f - g\|_{L^2}$. Provided that $\|g^l\|_{L^2}$ is bounded by $M/2$, we can conclude that

$$| |g^l|^2(E) - |g^m|^2(E) | \leq M \|g^l - g^m\|_{L^2}.$$

This inequality holds for all $E \in \mathcal{B}(\mathbb{R}^\infty)$ as in the proof of Proposition 2.4. Thus we obtain

$$\| |g^l|^2 - |g^m|^2 \|_M \leq M \|g^l - g^m\|_{L^2}.$$

According to the above inequality $|g^l|^2$ is a Cauchy sequence with respect to $\|\cdot\|$. From Lemma 2.7 there exist a positive Borel measure μ on \mathbb{R}^∞ as the limit of $|g^l|^2$ such that

$$\mu_n(E_n) = \int_{E_n} G_n(x) dx$$

for $G_n \in L^1(\mathbb{R}^n)$, $G_n(x) \geq 0$.

Set $g' = \{\sqrt{G_n}\}$ and $H_n(x) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} |g_{n+k}(x, x')|^2 dx'$. We see at once $g' \in \mathcal{L}_b^2(\mathbb{R}^\infty)$. In addition

$$\begin{aligned} \|g^l - [g']\|_{L^2}^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |g_n^l(x) - \sqrt{G_n(x)}|^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} ||g_n^l(x)|^2 - G_n(x)| dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} ||g_n^l(x)|^2 - H_n(x)| dx + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |H_n(x) - G_n(x)| dx \\ &\leq \|\mu - |g^l|^2\|. \end{aligned}$$

Thus we obtain $\lim_{l \rightarrow \infty} \|g^l - [g']\|_{\mathcal{L}^2} = 0$, which proves the theorem. □

Let us introduce two important operators on $L^2(\mathbb{R}^\infty)$: multiplication operators and convolution products. Suppose that $w \in L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), |f|^2)$ and $f \in L^2(\mathbb{R}^\infty)$. For arbitrary $g \in L^2(\mathbb{R}^\infty)$, let $T : L^2(\mathbb{R}^\infty) \rightarrow C$ be a linear functional defined by

$$T(g) = \int_{\mathbb{R}^\infty} w(x)(f \cdot g)(dx).$$

From (2.4) we have

$$|T(g)| \leq \left(\int_{\mathbb{R}^\infty} |w(x)|^2 |f|^2(dx) \right)^{\frac{1}{2}} \|g\|.$$

So T is bounded and there exists $h \in L^2(\mathbb{R}^\infty)$ such that $T(g) = \langle h, g \rangle$ because of Riesz representation theorem. We call h mentioned above the multiplication of w and f , and write $w \times f$.

Proposition 2.9. *Let $f, g \in L^2(\mathbb{R}^\infty)$ and u, v are bounded Borel measurable complex-valued functions on \mathbb{R}^∞ . Then, the following (1) – (5) hold.*

- (1) $(u + v) \times f = u \times f + v \times f$
- (2) $u \times (f + g) = u \times f + u \times g$
- (3) $\overline{u \times f} = \bar{u} \times \bar{f}$
- (4) $(u \times f, g) = (f, u \times g)$
- (5) $u \times (v \times f) = (u \times v) \times f$

Lemma 2.10. *Suppose that $f = \{f_n\}_{n=1}^\infty \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ and $\mu \in M_0(\mathbb{R}^\infty)$ be a positive measure such that the density function of $p_n(\mu)$ is given by $g_n \geq 0$ for all n . Then $\{f_n * g_n\} \in \mathcal{L}_b^2(\mathbb{R}^\infty)$ hold. Here $(f_n * g_n)(x) = \int_{\mathbb{R}^n} f_n(x - y)g_n(y)dy$.*

Proof. It suffices to show that $\{f_n * g_n\}$ satisfies (2.2). Hence by

$$\begin{aligned} & \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{n+1}(x-y, x'-y')|^2 g_{n+1}(y, y') dy' \right) dx' \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{n+1}(x-y, x'-y')|^2 dx' \right) g_{n+1}(y, y') dy' \\ &\leq |f_n(x-y)|^2 \int_{\mathbb{R}} g_{n+1}(y, y') dy' = |f_n(x-y)|^2 g_n(y), \end{aligned}$$

We have

$$\begin{aligned} & \int_{\mathbb{R}} |(f_{n+1} * g_{n+1})(x, x')|^2 dx' \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n+1}} f_{n+1}(x-y, x'-y') g_{n+1}(y, y') dy dy' \right)^2 dx' \\ &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}} |f_{n+1}(x-y, x'-y')|^2 g_{n+1}(y, y') dy'} \sqrt{\int_{\mathbb{R}} g_{n+1}(y, y') dy'} dy \right)^2 dx' \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \sqrt{\int_{\mathbb{R}} |f_{n+1}(x-s, x'-y')|^2 g_{n+1}(s, y') dy'} \sqrt{g_n(s)} \right. \\ &\quad \left. \times \sqrt{\int_{\mathbb{R}} |f_{n+1}(x-t, x'-y')|^2 g_{n+1}(t, y') dy'} \sqrt{g_n(t)} ds dt \right) dx' \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(\int_{\mathbb{R}} \sqrt{\int_{\mathbb{R}} |f_{n+1}(x-s, x'-y')|^2 g_{n+1}(s, y') dy'} \sqrt{g_n(s)} \right. \\ &\quad \left. \times \sqrt{\int_{\mathbb{R}} |f_{n+1}(x-t, x'-y')|^2 g_{n+1}(t, y') dy'} \sqrt{g_n(t)} dx' \right) ds dt \\ &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \sqrt{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{n+1}(x-s, x'-y')|^2 g_{n+1}(s, y') \right) dx'} \\ &\quad \times \sqrt{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{n+1}(x-t, x'-y')|^2 g_{n+1}(t, y') \right) dx'} \sqrt{g_n(s) g_n(t)} ds dt \\ &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \sqrt{|f_n(x-s)|^2 g_n(s)} \sqrt{|f_n(x-t)|^2 g_n(t)} \sqrt{g_n(s) g_n(t)} ds dt \\ &= |(f_n * g_n)(x)|^2, \end{aligned}$$

which proves the lemma. □

For arbitrary $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ and $\mu \in M_0(\mathbb{R}^\infty)$ such that the density function of $p_n(\mu)$ is given by g_n for all n , the convolution product of μ and f , denoted by $\mu * f$, is defined to be

$\{f_n * g_n\}$. Hence by Lemma 2.10, $\mu * f$ can be expressed as the complex linear combination of $\mathcal{L}_b^2(\mathbb{R}^\infty)$. It follows that $\mu * f \in \mathcal{L}^2(\mathbb{R}^\infty)$. In the case $f \in L^2(\mathbb{R}^\infty)$, choose $\{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ a representative of f and the convolution product of μ and f is defined by $\{\{f_n * g_n\}\}$. Since by Young's inequality, $\|\mu_n * f_n\| \leq \|\mu_n\| \|f_n\|$. By taking the limit of this inequality, we have $\|\mu * f\| \leq \|\mu\| \|f\|$. Thus the definition of $\mu * f$ is independent of the choice of the representative.

3 Fourier transform on $L^2(\mathbb{R}^\infty)$

Let $\mathbb{R}_0^\infty = \{(x_1, x_2, \dots) : \text{there exists } N, x_n = 0 \text{ for all } n > N\}$. The topology of \mathbb{R}_0^∞ is defined by seminorms $\{p_x\}_{x \in \mathbb{R}^\infty}$: $p_x(y) = \left| \sum_{n=1}^{\infty} x_n y_n \right|$ ($y \in \mathbb{R}_0^\infty$). Since there exists N such that $y_n = 0$ for $n > N$, the right hand side makes sense. \mathbb{R}_0^∞ is homeomorphic to the countable direct sum of \mathbb{R} .

Let $a = \{a_n\}$ be a positive sequence and H_a be a Hilbert space defined by $H_a = \{x = \{x_n\}_{n=1}^\infty; \sum_{n=1}^{\infty} a_n^2 x_n^2 < \infty\}$ equipped with the inner product $\langle x, y \rangle_a = \sum_{n=1}^{\infty} a_n^2 x_n y_n$ and the norm $\|x\|_a = \sqrt{\langle x, x \rangle_a}$.

The following assertions are special cases of Minlos' theorem and Sazanov' theorem. For a thorough treatment we refer the reader to [4], [28].

Theorem 3.1 (Minlos). *For all continuous function ϕ on \mathbb{R}_0^∞ satisfying*

$$\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \phi(x_i - x_j) \geq 0 \quad (3.1)$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}, x_1, \dots, x_n \in \mathbb{R}_0^\infty (n = 1, 2, \dots)$, there exists a bounded positive Borel measure μ on \mathbb{R}^∞ such that

$$\phi(x) = \int_{\mathbb{R}^\infty} e^{2\pi\sqrt{-1}\langle x, y \rangle} \mu(dy), \langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Such ϕ is called the characteristic function of μ , and denoted by $\hat{\mu}$. A function satisfying (3.1) is said to be positive definite. This theorem also holds for a continuous positive definite function on a nuclear space $S \subset \mathbb{R}^\infty$ as the characteristic function of the measure on its dual space S' . We will establish the relation between the inner product of $L^2(\mathbb{R}^\infty)$ and positive definite functions on \mathbb{R}^∞ .

Theorem 3.2 (Sazanov). *Suppose that $a = \{a_n\}$ and $b = \{b_n\}$ are positive sequences satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$. For all continuous positive definite function ϕ on H_b , there exists a positive bounded Borel measure μ on H_a whose characteristic function is given by ϕ .*

Set $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ and $h = (h_1, h_2, \dots) \in \mathbb{R}^\infty$. A translation operator τ_h on $\mathcal{L}^2(\mathbb{R}^\infty)$ is defined by $\tau_h f = \{f_n(x_1 - h_1, \dots, x_n - h_n)\}$. By definition, it follows that

$$\|\tau_h f\|_{\mathcal{L}^2} = \|f\|_{\mathcal{L}^2}. \quad (3.2)$$

Let $E \subset \mathbb{R}^\infty$ be a complete metric space. $f \in \mathcal{L}^2(\mathbb{R}^\infty)$ is said to be E -shift continuous if $\lim_{n \rightarrow \infty} \|\tau_{h_n} f - f\|_{\mathcal{L}^2} = 0$ whenever a sequence $\{h_n\}$ in E converges to limit 0. Let $S \subset \mathbb{R}^\infty$ be a nuclear space. $f \in \mathcal{L}^2(\mathbb{R}^\infty)$ is said to be S -shift continuous if $\lim_{\alpha} \|\tau_{h_\alpha} f - f\|_{\mathcal{L}^2} = 0$ whenever a net $\{h_\alpha\}$ in S converges to 0.

A translation operator τ_h on $L^2(\mathbb{R}^\infty)$ is defined by $\tau_h[f] = [\tau_h f]$ ($f \in \mathcal{L}^2(\mathbb{R}^\infty)$). Equation (3.2) makes this definition possible. Shift continuity of $L^2(\mathbb{R}^\infty)$ is defined as well as $\mathcal{L}^2(\mathbb{R}^\infty)$. We will denote by $\mathbb{L}^2(\mathbb{R}^\infty)$ the totality of \mathbb{R}_0^∞ -shift continuous elements of $L^2(\mathbb{R}^\infty)$.

Example 3.3. If a bounded positive Borel measure μ is of the form

$$\mu = \prod_{n=1}^{\infty} f_n dx_n, \quad f_n \in L^1(\mathbb{R}^n), \quad f_n \geq 0, \quad \int_{\mathbb{R}} f_n(x) dx = 1,$$

then the square root of μ is \mathbb{R}_0^∞ -shift continuous. In fact, letting $g_n = \mathcal{F}(\sqrt{f_n})$, $\int_{\mathbb{R}} |g_n(x)|^2 dx = 1$ follows by Parseval's equation. Thus $\lambda = \prod_{n=1}^{\infty} |g_n|^2 dx_n$ also defines a bounded positive Borel measure on \mathbb{R}^∞ . We obtain

$$\begin{aligned} \|\tau_h \sqrt{\mu} - \sqrt{\mu}\|_{\mathcal{L}^2}^2 &= \int_{\mathbb{R}^n} \left| \sqrt{f_1(x-h_1) \cdots f_n(x-h_n)} - \sqrt{f_1(x) \cdots f_n(x)} \right| dx_1 \cdots dx_n \\ &= \int_{\mathbb{R}^n} |e^{2\pi\sqrt{-1}\langle \xi, h \rangle_n} - 1|^2 |g_1(\xi)|^2 \cdots |g_n(\xi)|^2 d\xi_1 \cdots d\xi_n \\ &= \int_{\mathbb{R}^\infty} |e^{2\pi\sqrt{-1}\langle \xi, h \rangle} - 1|^2 \lambda(d\xi). \end{aligned}$$

Here $h = (h_1, \dots, h_n, 0, \dots)$. Because the characteristic function on \mathbb{R}^∞ is continuous function on \mathbb{R}_0^∞ , so is $\sqrt{\mu}$.

Definition 3.4. Assume that $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ is \mathbb{R}_0^∞ -shift continuous. Then, a Fourier transform of f is defined by $\{\hat{f}_n\}$ and denoted by \hat{f} or $\mathcal{F}f$. Here,

$$\hat{f}_n(\xi) = \text{l.i.m.}_{L \rightarrow \infty} \int_{|x|_n \leq L} e^{2\pi\sqrt{-1}\langle x, \xi \rangle_n} f(x) dx, \quad \langle x, \xi \rangle_n = \sum_{j=1}^n x_j \xi_j, \quad |x|_n = \sqrt{\langle x, x \rangle_n}.$$

The sequence $\{\hat{f}_n\}$ is not always included in $\mathcal{L}^2(\mathbb{R}^\infty)$. However, it is possible to give a definition of the square of $\hat{f} = \{\hat{f}_n\}$ as a bounded positive Borel measure.

Proposition 3.5. Assume that $f = \{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ is \mathbb{R}_0^∞ -shift continuous. Then, there exists a bounded positive Borel measure μ on \mathbb{R}^∞ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} e^{2\pi\sqrt{-1}\langle x, \xi \rangle_n} |\hat{f}_n(\xi)|^2 d\xi = \int_{\mathbb{R}^\infty} e^{2\pi\sqrt{-1}\langle x, y \rangle} \mu(dy) \quad (3.3)$$

for all $x \in \mathbb{R}_0^\infty$.

Proof. Set $\phi(x) = \langle \tau_x f, f \rangle_{\mathcal{L}^2}$ ($x \in \mathbb{R}_0^\infty$). By assumption, ϕ is continuous function on \mathbb{R}_0^∞ . And (3.2) ensures that

$$\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \phi(x_i - x_j) = \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \langle \tau_{x_i} f, \tau_{x_j} f \rangle_{\mathcal{L}^2} = \left\| \sum_{i=1}^n \alpha_i \tau_{x_i} f \right\|_{\mathcal{L}^2}^2 \geq 0.$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}, x_1, \dots, x_n \in \mathbb{R}_0^\infty$ ($n = 1, 2, \dots$). Thus ϕ is positive definite. From Minlos' theorem 3.1, there exists a bounded positive measure on \mathbb{R}^∞ such that

$$\langle \tau_x f, f \rangle_{\mathcal{L}^2} = \int_{\mathbb{R}^\infty} e^{2\pi\sqrt{-1}\langle x, y \rangle} \mu(dy).$$

By definition, the left hand side of this equation is equal to

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} f_n(y_1 - x_1, \dots, y_n - x_n) f_n(y_1, \dots, y_n) dy_1 \cdots dy_n \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} e^{2\pi\sqrt{-1}\langle x, y \rangle_n} |\hat{f}_n(y)|^2 dy, \end{aligned}$$

which proves the proposition. \square

Let μ_n^k be a bounded positive Borel measure on \mathbb{R}^n defined by

$$\mu_n^k(E) = \int_E \int_{\mathbb{R}^k} |\hat{f}_{n+k}(\xi, \xi')|^2 d\xi' d\xi$$

for $E \in \mathcal{B}(\mathbb{R}^n)$, and μ_n be a bounded positive Borel measure on \mathbb{R}^n defined by

$$\mu_n(E) = \mu(p_n^{-1}(E)). \quad (3.4)$$

Substituting $x = (x_1, \dots, x_n, 0, \dots) \in \mathbb{R}_0^\infty$ into (3.3) we can assert that the characteristic function of μ_n^k converges to that of μ_n at each point. Thus μ_n^k converges weakly to μ_n i.e.

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f(x) \mu_n^k(dx) = \int_{\mathbb{R}^n} f(x) \mu_n(dx)$$

for all continuous bounded functions f and $n = 1, 2, \dots$.

Let $\tilde{\mu}^k$ be a bounded positive Borel measure on \mathbb{R}^∞ such that

$$\tilde{\mu}^k(p_j^{-1}(E)) = \begin{cases} \mu_j^k(E) & j \leq k \\ \int_E |\hat{f}_k(\xi)|^2 \delta(\xi') d\xi d\xi' & j > k. \end{cases}$$

Here δ means the Dirac delta measure on \mathbb{R}^{j-k} .

At $\xi = (\xi_1, \dots, \xi_j, 0, \dots) \in \mathbb{R}_0^\infty (j \leq k)$, it follows that

$$\hat{\mu}^k(\xi_1, \dots, \xi_j, 0, \dots) = \hat{\mu}_j^k(\xi_1, \dots, \xi_j).$$

Thus the characteristic function of $\hat{\mu}^k$ converges to that of μ at each point. Since \mathbb{R}^∞ is nuclear space, it follows that $\hat{\mu}^k$ converges weakly to μ (see [4] for more details).

Proposition 3.6. *The measure μ defined in Proposition 3.5 satisfies*

$$\lim_{\alpha} \|\tau_{h_\alpha} \mu - \mu\| = 0 \quad (3.5)$$

whenever a net $\{h_\alpha\}$ in \mathbb{R}_0^∞ converges to limit 0.

We call a measure satisfying (3.5) the \mathbb{R}_0^∞ -shift continuous measure.

Proof. Fix $h \in \mathbb{R}_0^\infty$. Since $\tau_h \tilde{\mu}^k - \tilde{\mu}^k$ converges weakly to $\tau_h \mu - \mu$, it follows that

$$\|\tau_h \mu - \mu\| \leq \liminf_{k \rightarrow \infty} \|\tau_h \tilde{\mu}^k - \tilde{\mu}^k\| \quad (3.6)$$

(see for instance [24]). By definition, it follows that

$$\|\tau_h \tilde{\mu}^k - \tilde{\mu}^k\| = \int_{\mathbb{R}^k} \left| |\hat{f}_k(\xi_1 - h_1, \dots, \xi_k - h_k)|^2 - |\hat{f}_k(\xi_1, \dots, \xi_k)|^2 \right| d\xi_1 \cdots d\xi_k.$$

Since

$$\int_{\mathbb{R}^n} ||f(x)|^2 - |g(x)|^2| dx \leq (\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}) \|f - g\|_{L^2(\mathbb{R}^n)}$$

for all $f, g \in L^2(\mathbb{R}^n)$, it follows that

$$\begin{aligned} & \|\tau_h \tilde{\mu}^k - \tilde{\mu}^k\| \\ & \leq 2 \sqrt{\int_{\mathbb{R}^k} |\hat{f}_k(\xi)|^2 d\xi} \sqrt{\int_{\mathbb{R}^k} |\hat{f}_k(\xi_1 - h_1, \dots, \xi_k - h_k) - \hat{f}_k(\xi_1, \dots, \xi_k)|^2 d\xi_1 \cdots d\xi_k} \\ & = 2 \sqrt{\int_{\mathbb{R}^k} |f_k(x)|^2 dx} \sqrt{\int_{\mathbb{R}^k} |(1 - e^{2\pi\sqrt{-1}\langle x, h \rangle_k} f_k(x))|^2 dx}. \end{aligned}$$

At $h = (h_1, \dots, h_j, 0, \dots) \in \mathbb{R}_0^\infty$, it follows that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\mathbb{R}^k} |(1 - e^{2\pi\sqrt{-1}\langle x, h \rangle_k} f_k(x))|^2 dx \\ & = 2 \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} (1 - \cos 2\pi \langle x, h \rangle_n) \int_{\mathbb{R}^k} |f_{n+k}(x, x')|^2 dx' dx \\ & = 2 \int_{\mathbb{R}^\infty} (1 - \cos 2\pi \langle x, h \rangle) |f|^2(dx). \end{aligned}$$

Thus (3.6) is bounded by

$$4\|f\|_{L^2} \int_{\mathbb{R}^\infty} (1 - \cos 2\pi \langle x, h \rangle) |f|^2(dx) \quad (3.7)$$

at each $y \in \mathbb{R}_0^\infty$. Since (3.7) is the real part of the characteristic function of the square of f , it is continuous function on \mathbb{R}_0^∞ , which prove the theorem. \square

Proposition 3.5 implies that there exists a sequence of L^1 -functions $\{h_n\}$ which satisfies (2.5), and

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} g(x) \left(\int_{\mathbb{R}^k} |\hat{f}_{n+k}(x, x')|^2 dx' \right) dx = \int_{\mathbb{R}^n} g(x) h_n(x) dx \quad (3.8)$$

for all bounded continuous functions g on \mathbb{R}^n .

A deeper discussion make it possible to improve the convergence of (3.8). In fact, it follows that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left| h_n(x) - \int_{\mathbb{R}^k} |\hat{f}_{n+k}(x, x')|^2 dx' \right| dx = 0. \quad (3.9)$$

for $n = 1, 2, \dots$. Moreover, (3.9) is equivalent to the \mathbb{R}_0^∞ shift continuity of $f \in \mathcal{L}^2(\mathbb{R}^\infty)$. It shows that a totality of square roots of measures satisfying (3.9) forms a linear subspace of $\mathcal{L}^2(\mathbb{R}^\infty)$ and the squares of such ones are \mathbb{R}_0^∞ -shift continuous measures. A detailed proof will appear in elsewhere.

We call the measure defined by Proposition 3.5 the square of \hat{f} . For $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$, the product of \hat{f} and $\bar{\hat{g}}$ is defined by

$$\frac{|\hat{f} + \hat{g}|^2 - |\hat{f} - \hat{g}|^2 + \sqrt{-1}|\hat{f} + \sqrt{-1}\hat{g}|^2 - \sqrt{-1}|\hat{f} - \sqrt{-1}\hat{g}|^2}{4}$$

and denoted by $\hat{f} \cdot \bar{\hat{g}}$.

Proposition 3.7. Assume that $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$ and both of them are \mathbb{R}_0^∞ -shift continuous measures. Then we have

$$|\hat{f} \cdot \bar{\hat{g}}(E)| \leq \sqrt{|\hat{f}|^2(E)} \sqrt{|\hat{g}|^2(E)} \quad (3.10)$$

for all $E \in \mathcal{B}(\mathbb{R}^\infty)$.

Proof. Equation (3.9) ensures that

$$\hat{f} \cdot \bar{\hat{g}}(p_n^{-1}(E_n)) = \lim_{k \rightarrow \infty} \int_{E_n} \int_{\mathbb{R}^k} \hat{f}_{n+k}(\xi, \xi') \overline{\hat{g}_{n+k}(\xi, \xi')} d\xi' d\xi.$$

Thus we obtain 3.10 as in the proof of Proposition 2.4. \square

Let $\mathcal{FL}^2(\mathbb{R}^\infty)$ be the totality of the Fourier transform of \mathbb{R}_0^∞ -shift continuous square roots of measures. A sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{FL}^2} : \mathcal{FL}^2(\mathbb{R}^\infty) \times \mathcal{FL}^2(\mathbb{R}^\infty) \mapsto \mathbb{C}$ is defined by $\langle \hat{f}, \hat{g} \rangle_{\mathcal{FL}^2} = \hat{f} \cdot \bar{\hat{g}}(\mathbb{R}^\infty)$ and a seminorm is defined by $\|\hat{f}\|_{\mathcal{FL}^2} = \sqrt{\langle \hat{f}, \hat{f} \rangle_{\mathcal{FL}^2}}$. By definition, $\|\hat{f}\|_{\mathcal{FL}^2} = \|f\|_{\mathcal{L}^2}$ holds.

Let $f, g \in \mathcal{L}^2(\mathbb{R}^\infty)$ are \mathbb{R}_0^∞ -shift continuous measures. \hat{f} is said to be equivalent to \hat{g} if $\|\hat{f} - \hat{g}\|_{\mathcal{FL}^2} = 0$. It is easy to see that f is equivalent to g if and only if so are \hat{f} and \hat{g} . Let $\mathcal{FL}^2(\mathbb{R}^\infty)$ be the quotient set of $\mathcal{FL}^2(\mathbb{R}^\infty)$ by this equivalence relation.

A Fourier transform of $f \in \mathbb{L}^2(\mathbb{R}^\infty)$ is defined by the equivalence class of the Fourier transform of a representative element of f , and denoted by \hat{f} . By definition, the Fourier transform maps $\mathbb{L}^2(\mathbb{R}^\infty)$ to $\mathcal{FL}^2(\mathbb{R}^\infty)$. Thus $\mathcal{FL}^2(\mathbb{R}^\infty)$ is a Hilbert space. From above, our Fourier transform is formulated as a unitary operator between two Hilbert spaces.

Suppose that $w \in L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), |\hat{f}|^2)$ and $f \in \mathcal{FL}^2(\mathbb{R}^\infty)$. We call the multiplication of w and \hat{f} to be a linear functional satisfying

$$\langle w \times \hat{f}, \hat{g} \rangle = \int_{\mathbb{R}^\infty} w(\xi) (\hat{f} \cdot \bar{\hat{g}})(d\xi)$$

for arbitrary $g \in \mathbb{L}^2(\mathbb{R}^\infty)$.

By applying Minlos' and Sazanov' theorems, we are able to discuss the support of the Fourier transform of the square roots of measures.

Theorem 3.8. *Let $S \subset \mathbb{R}^\infty$ be a nuclear space and $f \in L^2(\mathbb{R}^\infty)$ be a S -shift continuous square root of a measure. It follows that*

$$|\hat{f}|^2(\mathbb{R}^\infty \setminus S') = 0.$$

where S' is the dual space of S .

Theorem 3.9. *Suppose that $a = \{a_n\}$ and $b = \{b_n\}$ are positive sequences satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$. If $f \in L^2(\mathbb{R}^\infty)$ be a H_a -shift continuous square root of a measure, then it follows that $|\hat{f}|^2(\mathbb{R}^\infty \setminus H_a) = 0$.*

Let $f \in L^2(\mathbb{R}^\infty)$ be a H_a -shift continuous square root of a measure and $\rho \in R_0^\infty$. Since

$$\langle \mathcal{F}(\tau_\rho f), \hat{g} \rangle = \int_{\mathbb{R}^\infty} e^{2\pi\sqrt{-1}\langle \xi, \rho \rangle} (\hat{f} \cdot \bar{\hat{g}})(d\xi)$$

for arbitrary $g \in \mathbb{L}^2(\mathbb{R}^\infty)$, we have $\mathcal{F}(\tau_\rho f) = e^{2\pi\sqrt{-1}\langle \cdot, \rho \rangle} \times \hat{f}$. Let $a = \{a_n\}$ and $b = \{b_n\}$ are positive sequences satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$ and $f \in L^2(\mathbb{R}^\infty)$ be a H_a -shift continuous square root of a measure. For arbitrary $\rho \in H_a$, let $\{\rho_k\}_{k=1}^\infty \subset \mathbb{R}_0^\infty$ be a sequence such that $\lim_{k \rightarrow \infty} \|\rho_k - \rho\|_{H_a} = 0$. Since $|\hat{f}|^2(\mathbb{R}^\infty \setminus H_a) = 0$ by theorem 3.9, we have

$$\|\tau_{\rho_m} f - \tau_{\rho_n} f\|^2 = \int_{H_b} |e^{2\pi\sqrt{-1}\langle \xi, \rho_m \rangle} - e^{2\pi\sqrt{-1}\langle \xi, \rho_n \rangle}|^2 |\hat{f}|^2(d\xi).$$

Because f is the H_a -shift continuous square root of a measure, $\|\tau_{\rho_m} f - \tau_{\rho_n} f\|^2 \rightarrow 0$ as $m \rightarrow \infty, n \rightarrow \infty$. This shows that $e^{2\pi\sqrt{-1}\langle \cdot, \rho_n \rangle}$, $n = 1, 2, \dots$ is a Cauchy sequence in the norm topology

of $L^2(H_b, \mathcal{B}(H_b), |\hat{f}|^2)$. We call the limit of these functions as $E(\rho) \in L^2(H_b, \mathcal{B}(H_b), |\hat{f}|^2)$. From this notation, we have

$$\mathcal{F}(\tau_\rho f) = E(\rho) \times \hat{f}.$$

It is easy to check $E(\rho_1 + \rho_2) = E(\rho_1)E(\rho_2)$ for all $\rho_1, \rho_2 \in H_a$.

Proposition 3.10. *Let $a = \{a_n\}$ be a positive sequence, μ be a complex Borel measure on H_a and $f \in L^2(\mathbb{R}^\infty)$ be H_a -shift continuous. We have*

$$\int_{H_b} (\tau_\rho f) \mu(d\rho) = \mu * f. \quad (3.11)$$

Here the left hand side is defined via the Bochner integral.

Proof. It follows immediately that

$$\left\langle \int_{H_b} (\tau_\rho f) \mu(d\rho), g \right\rangle = \int_{H_a} \langle \tau_\rho f, g \rangle \mu(d\rho).$$

On the other hand, assuming that $\{f_n\}$ and $\{g_n\}$ are a representative of f and g respectively, we have

$$\langle \mu * f, g \rangle = \lim_{n \rightarrow \infty} \int_{H_a} \langle \tau_{p_n(\rho)} f_n, g_n \rangle_{L^2(\mathbb{R}^n)} \mu(d\rho) = \int_{H_a} \langle \tau_\rho f, g \rangle \mu(d\rho).$$

These two equations follow for arbitrary $g \in L^2(\mathbb{R}^\infty)$, (3.11) is proved. \square

The Fourier transform of convolution product $\mu * f$ is equal to the multiplication of the characteristic function μ to the Fourier transform of f .

Theorem 3.11. *Suppose that $a = \{a_n\}$ and $b = \{b_n\}$ are positive sequences satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$. Let μ be a complex Borel measure on H_a such that the characteristic function $\hat{\mu}$ is continuously extendable over H_b and $f \in L^2(\mathbb{R}^\infty)$ be H_a -shift continuous. Then we have*

$$\mathcal{F}(\mu * f) = \hat{\mu} \times \hat{f}.$$

Proof. By equation (3.11), we have

$$\begin{aligned} \mathcal{F} \left(\int_{H_b} (\tau_\rho f) \mu(d\rho) \right) &= \int_{H_b} \mathcal{F}(\tau_\rho f) \mu(d\rho) = \int_{H_b} E(\rho) \times \hat{f} \mu(d\rho) \\ &= \int_{H_b} E(\rho) \mu(d\rho) \times \hat{f} = \hat{\mu} \times \hat{f}. \end{aligned}$$

\square

4 differential calculus for square roots of measure

In Averbuh-Smolyanov-Fomin [2], the differentiation (in the sense of Fomin) of complex measures is formulated as follows.

Definition 4.1. Let $\mu \in M(\mathbb{R}^\infty)$ and $\rho \in H_a$. μ is called differentiable in the direction ρ if

$$\lim_{t \rightarrow 0} \left\| \frac{\tau_{t\rho}\mu - \mu}{t} - \lambda \right\| = 0$$

for some $\lambda \in M(X)$. λ is denoted by $\partial_\rho \mu$ and called the directional derivative of μ in the direction ρ .

Definition of the differentiation for square roots of measures is parallel to the case of complex measures.

Definition 4.2. Let $f \in L^2(\mathbb{R}^\infty)$ be a H_a -shift continuous square root of a measure and $\rho \in H_a$. f is called differentiable in the direction ρ if

$$\lim_{t \rightarrow 0} \left\| \frac{\tau_{t\rho}f - f}{t} - g \right\| = 0$$

for some $g \in L^2(\mathbb{R}^\infty)$. g is denoted by $\partial_\rho f$ and called the directional derivative of f in the direction ρ . If $\partial_\rho f$ exists for all $\rho \in H_a$, f is called H_a differentiable.

This two differentiation is related with the following chain-rule. By using this, we are able to translate the properties of differentiation for the square roots of measures into those for the ordinary measures, or vice versa.

Proposition 4.3 (chain-rule). Let $f, g \in L^2(\mathbb{R}^\infty)$ are H_a -shift continuous square roots of measures and $\rho \in H_a$. If f, g are differentiable in the direction ρ , the product measure $f \cdot g$ are also differentiable in the direction h , and it follows that

$$\partial_\rho(f \cdot g) = \partial_\rho f \cdot g + f \cdot \partial_\rho g.$$

Since $\partial_\rho(f \cdot g)(\mathbb{R}^\infty) = 0$, we also have

$$\langle \partial_\rho f, g \rangle = -\langle f, \partial_\rho g \rangle.$$

Proof. Let $\rho \in H_a$ and $f, g \in L^2(\mathbb{R}^\infty)$ are differentiable square roots of measures in the direction ρ . We have

$$\begin{aligned} & \left\| \frac{\tau_{t\rho}(f \cdot g) - (f \cdot g)}{t} - \partial_\rho f \cdot g - f \cdot \partial_\rho g \right\| = \left\| \tau_{t\rho}f \cdot \frac{\tau_{t\rho}g - g}{t} - f \cdot \partial_\rho g \right\| + \left\| \frac{\tau_{t\rho}f - f}{t} \cdot g - \partial_\rho f \cdot g \right\| \\ & \leq \|\tau_{t\rho}f - f\| \left\| \frac{\tau_{t\rho}g - g}{t} \right\| + \|f\| \left\| \frac{\tau_{t\rho}g - g}{t} - \partial_\rho g \right\| + \|g\| \left\| \frac{\tau_{t\rho}f - f}{t} - \partial_\rho f \right\| \rightarrow 0 \text{ as } t \rightarrow 0 \end{aligned}$$

by Schwarz inequality (2.7). This finishes the proof. \square

Suppose that $a = \{a_n\}$ and $b = \{b_n\}$ are positive sequences satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$. Let $\rho \in H_a$ and $f \in L^2(\mathbb{R}^\infty)$ be a differentiable square root of a measure in the direction ρ . Since

$$\left\| \frac{\tau_{t\rho}f - f}{t} - \partial_\rho f \right\| = \left\| \frac{E(t\rho) - 1}{t} \times \hat{f} - \mathcal{F}(\partial_\rho f) \right\| \rightarrow 0 \text{ as } t \rightarrow 0,$$

and $|\hat{f}|^2(R^\infty \setminus H_b) = 0$, we have

$$\left\| \frac{E(s\rho) - 1}{s} \times \hat{f} - \frac{E(t\rho) - 1}{t} \times \hat{f} \right\|^2 = \int_{H_b} \left| \frac{E(s\rho) - 1}{s} - \frac{E(t\rho) - 1}{t} \right|^2 d|\hat{f}|^2 \rightarrow 0 \text{ as } s, t \rightarrow 0.$$

Thus there exists $W(\rho) \in L^2(H_b, \mathcal{B}(H_b), |\hat{f}|^2)$ such that $\lim_{t \rightarrow 0} \|t^{-1}(E(t\rho) - 1) - 2\pi\sqrt{-1}W(\rho)\| = 0$. This shows that

$$\mathcal{F}(\partial_\rho f) = 2\pi\sqrt{-1}W(\rho) \times \hat{f}.$$

We see at once that

- (1) $W(\rho_1 + \rho_2) = W(\rho_1) + W(\rho_2)$ for all $\rho_1, \rho_2 \in H_a$,
- (2) $W(\alpha\rho) = \alpha W(\rho)$ for all $\alpha \in \mathbb{C}, \rho \in H_a$,
- (3) $W(\rho) = \langle \xi, \rho \rangle$ for all $\rho \in R_0^\infty$,
- (4) $E(\rho) = e^{2\pi\sqrt{-1}W(\rho)}$ for all $\rho \in H_a$.

Proposition 4.4. *Let $f \in L^2(\mathbb{R}^\infty)$ be a differentiable square root of a measure in the direction $\rho \in H_a$ and $A \in \mathcal{B}(\mathbb{R}^\infty)$. Then $|f|^2(A) = 0$ implies that $|\partial_\rho f|^2(A) = 0$. So $|\partial_\rho f|^2$ is absolutely continuous with respect to $|f|^2$.*

Proof. Since $|f|^2(A) = 0$ and $(f \cdot \tau_{t\rho}f)(A) = 0$, we have

$$|\partial_\rho f|^2(A) = \lim_{t \rightarrow 0} \left| \frac{\tau_{t\rho}f - f}{t} \right|^2(A) = \left(\frac{d}{dt} \sqrt{|f|^2(A + t\rho)} \right)^2 \Big|_{t=0}.$$

Because $|f|^2(A + t\rho)$ attains minimum value 0 when $t = 0$, derivative of $\sqrt{|f|^2(A + t\rho)}$ at $t = 0$ is equal to 0. Thus we have $|\partial_\rho f|^2(A) = 0$. \square

Proposition 4.5. *Let $f \in L^2(\mathbb{R}^\infty)$ be a differentiable square root of a measure in the direction $\rho \in H_a$. Then $\partial_\rho f$ is also H_a -shift continuous.*

Proof. For $\sigma \in \mathbb{R}_0^\infty$ we have

$$\|\tau_\sigma(\partial_\rho f) - \partial_\rho f\|^2 = \int_{\mathbb{R}^\infty} |e^{2\pi\sqrt{-1}W(\sigma)} - 1|^2 |W(\rho)|^2 d|\hat{f}|^2.$$

Since $W(\rho) \in L^2(H_b, \mathcal{B}(H_b), |\hat{f}|^2)$ ($b = \{b_n\}$ is a positive sequence satisfying $\sum_{n=1}^{\infty} a_n^2 b_n^2 < \infty$), for all $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{|W(\rho)| > R} |W(\rho)|^2 d|\hat{f}|^2 < \epsilon/2.$$

Thus we obtain

$$\begin{aligned} \|\tau_\sigma(\partial_\rho f) - \partial_\rho f\|^2 &= \left(\int_{|W(\rho)| \leq R} + \int_{|W(\rho)| > R} \right) |e^{2\pi\sqrt{-1}W(\sigma)} - 1|^2 |W(\rho)|^2 d|\hat{f}|^2 \\ &\leq R^2 \int_{\mathbb{R}^\infty} |e^{2\pi\sqrt{-1}W(\sigma)} - 1|^2 d|\hat{f}|^2 + 2 \int_{|W(\rho)| > R} |W(\rho)|^2 d|\hat{f}|^2 \leq R^2 \|\tau_\sigma f - f\|^2 + \epsilon. \end{aligned}$$

Since $\|\tau_\sigma f - f\|^2 \rightarrow 0$ as $\|\sigma\|_{H_a} \rightarrow 0$, we have $\lim_{\|\sigma\|_{H_a} \rightarrow 0} \|\tau_\sigma(\partial_\rho f) - \partial_\rho f\|^2 \leq \epsilon$ for all $\epsilon > 0$. It means that $\lim_{\|\sigma\|_{H_a} \rightarrow 0} \|\tau_\sigma(\partial_\rho f) - \partial_\rho f\|^2 = 0$. \square

Definition 4.6. (*Fréchet derivative*) Let $f \in L^2(\mathbb{R}^\infty)$ be H_a -shift continuous. If there is a bounded linear operator $df : H_a \mapsto L^2(\mathbb{R}^\infty)$ such that

$$\lim_{\|\rho\|_{H_a} \rightarrow 0} \|\tau_\rho f - f - df(\rho)\| / \|\rho\|_{H_a} = 0,$$

f is said to be Fréchet differentiable in the direction of H_a and df is called H_a Fréchet derivative in the direction of H_a . Moreover if there is a bounded bilinear form $d^2 f : H_a \times H_a \mapsto L^2(\mathbb{R}^\infty)$ such that

$$\lim_{\|\rho_2\|_{H_a} \rightarrow 0} \|\tau_{\rho_2}(df(\rho_1)) - df(\rho_1) - d^2 f(\rho_1, \rho_2)\| / \|\rho_2\|_{H_a} = 0$$

for all $\rho_1 \in H_a$, f is said to be twice Fréchet differentiable in the direction of H_a and $d^2 f$ is called second order Fréchet derivative in the direction of H_a .

The following theorem gives a sufficient condition for the existence of H_a Fréchet derivatives of square roots of measures.

Lemma 4.7. Let $f \in L^2(\mathbb{R}^\infty)$ be a real-valued R_0^∞ -shift continuous square root of a measure. Then

$$\int_{\mathbb{R}^\infty} \xi_i \xi_j d|\hat{f}|^2 = 0, \text{ for all } i, j \in N, i \neq j.$$

Proof. For simplicity we take $i = 1$ and $j = 2$. Let $\{f_n\} \in \mathcal{L}^2(\mathbb{R}^\infty)$ be a representative of f . Since f_n ($n = 1, 2, \dots$) are real-valued functions, it follows that $\hat{f}_n(\xi) = \hat{f}_n(-\xi)$, and we have

$$\begin{aligned} \int_{\mathbb{R}^\infty} \xi_1 \xi_2 d|\hat{f}|^2 &= \lim_{k \rightarrow \infty} \int_{R^{2+k}} \xi_1 \xi_2 |\hat{f}(\xi_1, \xi_2, \xi')|^2 d\xi_1 d\xi_2 d\xi' \\ &= \lim_{k \rightarrow \infty} \int_R \xi_1 \left(\int_{R^{1+k}} \xi_2 \hat{f}(\xi_1, \xi_2, \xi') \hat{f}(-\xi_1, -\xi_2, -\xi') d\xi_2 d\xi' \right) d\xi_1. \end{aligned}$$

Because the integrand is an odd function with respect to ξ_1 for all $k \in N$, the right hand side is equal to 0. \square

Theorem 4.8. *Let $f \in L^2(\mathbb{R}^\infty)$ be a real-valued H_a differential square root of a measure. It follows that*

$$\|\partial_\rho f\|^2 \leq \left(\sup_{n \geq 1} \int_{\mathbb{R}^\infty} a_n^{-2} \xi_n^2 d|\hat{f}|^2 \right) \|\rho\|_{H_a}^2,$$

for all $\rho \in H_a$ and if $\sup_{n \geq 1} \int_{\mathbb{R}^\infty} a_n^{-2} \xi_n^2 d|\hat{f}|^2 < \infty$, the mapping $\rho \mapsto \partial_\rho f$ is a one-to-one linear bounded operator from H_a to $L^2(\mathbb{R}^\infty)$.

Proof. Let $\rho \in R_0^\infty$. By lemma 4.7 we have

$$\begin{aligned} \|\partial_\rho f\|^2 &= \int_{\mathbb{R}^\infty} |W(\rho)|^2 d|\hat{f}|^2 = \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} \rho_k^2 \xi_k^2 d|\hat{f}|^2 + 2 \sum_{i>j} \int_{\mathbb{R}^\infty} \rho_i \rho_j \xi_i \xi_j d|\hat{f}|^2 \\ &= \sum_{k=1}^{\infty} a_k^2 \rho_k^2 \int_{\mathbb{R}^\infty} a_k^{-2} \xi_k^2 d|\hat{f}|^2 \leq \left(\sup_{n \geq 1} \int_{\mathbb{R}^\infty} a_n^{-2} \xi_n^2 d|\hat{f}|^2 \right) \|\rho\|_{H_a}^2. \end{aligned}$$

Since $\rho \mapsto \partial_\rho f$ is a linear mapping from H_a to $L^2(\mathbb{R}^\infty)$, it is continuously extendable uniquely over H_a . \square

We write B_a for Banach space $\{\xi \in \mathbb{R}^\infty : \sup_{n \geq 1} |a_n^{-1} \xi_n| < \infty\}$, and write $\|\xi\|_{B_a} = \sup_{n \geq 1} |a_n^{-1} \xi_n|$.

If $\sum_{n=1}^{\infty} a_n b_n < \infty$ for positive numbers $\{b_n\}$, we have $B_a \subset H_b$ because

$$\|\xi\|_{H_b}^2 = \sum_{n=1}^{\infty} a_n^2 b_n^2 a_n^{-2} \xi_n^2 \leq \|\xi\|_{B_a}^2 \sum_{n=1}^{\infty} a_n b_n.$$

5 Lévy Laplacian and Fourier transform

In this section we introduce the Lévy Laplacian for square roots of measures on a classical Wiener space $C_0[0, T]$, the space of all real-valued continuous functions on $[0, T]$ which start at the origin. To do this, we will give an embedding of $C_0[0, T]$ into \mathbb{R}^∞ in the following way.

For $s \in \mathbb{R}$, let

$$h_s = \{a = \{a_n\}_{n=1}^\infty \in R^\infty : \sum_{n=1}^{\infty} (1 + n^2)^s |a_n|^2 < \infty\}.$$

h^s is a Hilbert space equipped with the inner product and the norm

$$\langle a, b \rangle_s = \sum_{n=1}^{\infty} (1 + n^2)^s a_n b_n \quad \|a\|_s^2 = \sum_{n=1}^{\infty} (1 + n^2)^s a_n^2 \quad a, b \in h_s,$$

and let $\mathbf{s} = \bigcap_{n=1}^{\infty} h^n$ be a locally convex topological vector space equipped with the seminorms $\|\cdot\|_s$ ($s \in R$). \mathbf{s} is called the space of rapidly decreasing sequence and it is known to be a nuclear space. Let \mathbf{s}' be a dual space of \mathbf{s} whose duality is given by the bilinear form

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n b_n, \quad a \in \mathbf{s}, \quad b \in \mathbf{s}'.$$

Let $\{e_n\}$ be a complete orthonormal system of $H_0^1[0, T] = \{\phi : \phi, \phi' \in L^2[0, T], \phi(0) = 0\}$

$$e_0(s) = \frac{s}{\sqrt{T}}, \quad e_n(s) = \frac{\sqrt{2T}}{n\pi} \sin \frac{n\pi s}{T}, \quad n = 1, 2, \dots,$$

and for $a = \{a_n\} \in \mathbf{s}'$, we make one-to-one correspondence $a \mapsto \phi(s) = \sum_{n=1}^{\infty} a_n e_n(s)$ from \mathbf{s}' to $\mathcal{D}'[0, T]$: totality of distributions on $[0, T]$. We also call the image of \mathbf{s} as $D[0, T]$.

Via this correspondence, we identify the Borel subset of \mathbf{s}' with the Borel subset of $\mathcal{D}'[0, T]$ such as

$$\mathcal{B}(\mathbf{s}') \ni A \mapsto \{\phi \in \mathcal{D}'[0, T] : (\langle \phi, e_n \rangle, \langle \phi, e_2 \rangle, \dots) \in A\} \in \mathcal{B}(\mathcal{D}'[0, T]).$$

Thus we are able to identify square roots of measures on $\mathcal{D}'[0, T]$ with those on \mathbf{s}' . In this sense we will denote by $L^2(D'[0, T])$ the set of the square roots of measures on $D'[0, T]$ as the image of $L^2(\mathbf{s})$. The notion of the differentiation, the multiplication operator, and the convolution product are defined in the same manner. As for the Fourier transform, if f be a \mathbf{s} -shift continuous square root of measure on \mathbf{s}' , then $|\hat{f}|^2(\mathbb{R}^\infty \setminus \mathbf{s}') = 0$ by theorem 3.8. Thus if $f \in L^2(D'[0, T])$ is a $D[0, T]$ -shift continuous square root of a measure, then $|\hat{f}|^2$ is defined as a positive bounded Borel measure on $D'[0, T]$.

Note that for $\phi \in C_0[0, T]$ we have

$$\langle \phi, e_n \rangle = \sqrt{\frac{2}{T}} (-1)^k \phi(T) + \sqrt{\frac{2}{T}} \int_0^\pi \phi\left(\frac{T}{\pi} s\right) k \sin k s ds.$$

Definition 5.1. (*Lévy-Laplacian*) Let $f \in L^2(D'[0, T])$ be $H_0^1[0, T]$ -shift continuous and f is twice differentiable in the direction e_n for all n . If $\frac{1}{n} \sum_{k=0}^n \partial_{e_k}^2 f$ approaches to some $g \in L^2(D'[0, T])$ in the norm topology of $L^2(D'[0, T])$, we write $g = \Delta_L f$ and the operator Δ_L is called the Lévy Laplacian for square roots of measures with respect to the CONS $\{e_n\}_{n=0}^\infty$.

We will examine the domain of the Lévy Laplacian for square roots of measures by using Fourier transform. If f belongs to the domain of Lévy Laplacian, we will have

$$\mathcal{F}(\Delta_L f) = - \lim_{n \rightarrow \infty} 4\pi^2 \frac{\langle \phi, e_0 \rangle^2 + \dots + \langle \phi, e_n \rangle^2}{n} \times \hat{f}$$

when the limit of the right hand side exists. So we first provide a sufficient condition for existence of the limit of $\frac{1}{n} \sum_{k=0}^{\infty} \langle \phi, e_k \rangle^2$.

Definition 5.2. Let Δ be a partition of the interval $[0, T] : 0 = v_0 < v_1 < \cdots < v_{n-1} < v_n = T$, $|\Delta| = \max_{1 \leq i \leq n} |v_i - v_{i-1}|$. We call

$$Q_T(\phi, \Delta) = \sum_{i=1}^n |\phi(v_i) - \phi(v_{i-1})|^2$$

the quadratic variation of $\phi \in C_0[0, T]$ over $[0, T]$ with partition Δ . If $Q_T(\phi, \Delta)$ converges to some limit as $|\Delta| \rightarrow 0$, we call this the quadratic variation of $\phi \in C_0[0, T]$ over $[0, T]$ and write $\langle \phi \rangle_T$.

Theorem 5.3. If there exists the quadratic variation of $\phi \in C_0[0, T]$ over $[0, T]$, it follows that

$$\lim_{n \rightarrow \infty} \frac{\langle \phi, e_0 \rangle^2 + \cdots + \langle \phi, e_n \rangle^2}{n} = \frac{\langle \phi \rangle_T}{T}.$$

To prove this theorem, we make use of the theory of summability method.

Definition 5.4. Let $\sum_{n=0}^{\infty} u_n$ be a real-valued series such that $f(x) = \sum_{n=0}^{\infty} u_n x^n$ converges for all

$0 \leq x < 1$. If $f(x)$ converges to a some limit s as $x \rightarrow 1 - 0$, we say that $\sum_{n=0}^{\infty} u_n$ Abel converges

to s and write $A - \sum_{n=0}^{\infty} u_n = s$. Let $s_n = a_1 + \cdots + a_n (n = 0, 1, \cdots)$, $f(x)$ is also written by

$$f(x) = (1 - x) \sum_{n=0}^{\infty} s_n x^n \quad (0 \leq x < 1).$$

If $\frac{1}{n} \sum_{k=0}^n s_k$ converges to a some limit s as $n \rightarrow \infty$, we say that $\sum_{n=0}^{\infty} u_n$ Cesàro converges to s and

write $(C, 1) - \sum_{n=0}^{\infty} u_n = s$.

Proposition 5.5. (Hardy-Littlewood) If $A - \lim_{n \rightarrow \infty} s_n = s$ and $s_n \geq 0$ for all n , then $(C, 1) - \lim_{n \rightarrow \infty} s_n = s$.

proof of theorem 5.3: Proposition 5.5 shows that

$$\lim_{n \rightarrow \infty} \frac{\langle \phi, e_0 \rangle^2 + \cdots + \langle \phi, e_n \rangle^2}{n} = \lim_{x \rightarrow 1-0} (1 - x) \sum_{n=0}^{\infty} \langle \phi, e_n \rangle^2 x^n$$

if the limit of the right hand side exists. Remember that

$$\langle \phi, e_n \rangle = \sqrt{\frac{2}{T}} (-1)^k \phi(T) + \sqrt{\frac{2}{T}} \int_0^\pi \phi\left(\frac{T}{\pi} s\right) k \sin k s ds$$

for all $\phi \in C_0[0, T]$. For $0 \leq x < 1$ we have

$$(1-x) \sum_{n=0}^{\infty} \langle \phi, e_n \rangle^2 x^n = \frac{2}{T} \left\{ \phi^2(T) + 2(1-x) \int_0^{\pi} \sum_{k=0}^{\infty} (-1)^k \phi(T) \phi\left(\frac{T}{\pi}s\right) kx^k \sin ksd s \right. \\ \left. + (1-x) \sum_{k=0}^{\infty} x^k \left(\int_0^{\pi} \phi\left(\frac{T}{\pi}s\right) k \sin ksd s \right)^2 \right\}.$$

We write $I_1(x)$ for the second term and $I_2(x)$ for the third term. Let $P_0(x, s)$ be the Poisson kernel and $P_1(x, s)$ be as follows.

$$P_0(x, s) = \frac{1}{\pi} \frac{1-x^2}{1-2x \cos s + x^2} \quad (0 \leq x \leq 1, 0 \leq s \leq \pi) \\ P_1(x, s) = -(1-x) \frac{\partial}{\partial s} P_0(x, s) \quad (0 \leq x \leq 1, 0 \leq s \leq \pi).$$

Since by

$$\sum_{k=0}^{\infty} kx^k \sin ks = -\frac{\partial}{\partial s} \left(\frac{1}{2} + \sum_{k=0}^{\infty} x^k \cos ks \right) = -\frac{\pi}{2} \frac{\partial}{\partial s} P_0(x, s),$$

we have

$$I_1(x) = -\frac{4}{T} (1-x) \phi(T) \int_0^{\pi} \phi\left(\frac{T}{\pi}s\right) \sum_{k=0}^{\infty} kx^k \sin k(\pi-s) ds \\ = -\frac{2\pi}{T} \phi(T) \int_0^T P_1(x, \pi-s) \phi\left(\frac{T}{\pi}s\right) ds$$

Lemma 5.6. For all $\phi \in C[0, \pi]$, we have

$$\lim_{x \rightarrow 1-0} \int_0^{\pi} P_1(x, s) \phi(s) ds = \frac{2}{\pi} \phi(0).$$

Proof. Since

$$P_1(x, s) = \frac{2}{\pi} x(1-x)(1-x^2) \frac{\sin s}{(1-2x \cos s + x^2)^2},$$

we have $P_1(x, s) \geq 0$ ($0 \leq s \leq \pi$, $0 \leq x \leq 1$). Then we have

$$\int_0^{\pi} P_1(x, s) ds = [-(1-x)P_0(x, s)]_0^{\pi} = (1-x)P_0(x, 0) - (1-x)P_0(x, \pi) \\ = \frac{1}{\pi}(1+x) - \frac{1}{\pi} \frac{(1-x)^2}{1+x} \rightarrow \frac{2}{\pi} \quad (x \rightarrow 1-0).$$

Hence by

$$\int_{\delta}^{\pi} P_1(x, s) ds = (1-x) \{P_0(x, \delta) - P_0(x, 0)\} \rightarrow 0 \quad (x \rightarrow 1-0) \quad (5.1)$$

for all $\delta > 0$, we obtain

$$\begin{aligned} & \left| \int_0^\pi P_1(x, s) \{ \phi(s) - \phi(0) \} ds \right| \\ & \leq \sup_{0 < s < \delta} |\phi(s) - \phi(0)| \int_0^\delta P_1(x, s) ds + 2 \sup_{0 < s < \pi} |\phi(s)| \int_\delta^\pi P_1(x, s) ds \end{aligned}$$

for all $\delta > 0$ and $\phi \in C[0, \pi]$. Since ϕ is uniformly continuous, by choosing δ small enough the first term converges to 0 as $x \rightarrow 1 - 0$. Because of (5.1), the second term also converges to 0 as $x \rightarrow 1 - 0$. \square

Thus we have $\lim_{x \rightarrow 1-0} I_1(x) = -\frac{4}{T} \phi^2(T)$. For $I_2(x)$ we have

$$\begin{aligned} I_2(x) &= \frac{2}{T} (1-x) \sum_{k=1}^{\infty} x^k \int_0^\pi \int_0^\pi \phi\left(\frac{T}{\pi}s\right) \phi\left(\frac{T}{\pi}t\right) k^2 \sin ks \sin ktdsdt \\ &= -\frac{1}{T} (1-x) \int_0^\pi \int_0^\pi \left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 \sum_{k=0}^{\infty} k^2 x^k \sin ks \sin ktdsdt \\ &\quad + \frac{1}{T} (1-x) \int_0^\pi \int_0^\pi \left\{ \phi^2\left(\frac{T}{\pi}s\right) + \phi^2\left(\frac{T}{\pi}t\right) \right\} \sum_{k=0}^{\infty} k^2 x^k \sin ks \sin ktdsdt. \end{aligned}$$

We write $J_1(x)$ for the first term and $J_2(x)$ for the second term. Hence by Lemma 5.6 and

$$\begin{aligned} J_2(x) &= \frac{2}{T} \int_0^\pi \phi^2\left(\frac{T}{\pi}s\right) \sum_{k=0}^{\infty} \{(-1)^k - 1\} k x^k \sin ks ds \\ &= \frac{\pi}{T} \int_0^\pi \{P_1(x, \pi - s) + P_1(x, s)\} \phi^2\left(\frac{T}{\pi}s\right) ds, \end{aligned}$$

we have $\lim_{x \rightarrow 1-0} J_2(x) = \frac{2}{T} \phi^2(T)$. Taken together, if $\lim_{x \rightarrow 1-0} J_1(x)$ exists, we will have

$$\lim_{x \rightarrow 1-0} (1-x) \sum_{n=0}^{\infty} \langle \phi, e_n \rangle^2 x^n = \lim_{x \rightarrow 1-0} J_1(x).$$

So we will concerned with the limit of $J_1(x)$ as $x \rightarrow 1 - 0$. Here since

$$\begin{aligned} (1-x) \sum_{k=0}^{\infty} k^2 x^k \sin ks \sin kt &= \frac{1}{2} (1-x) \sum_{k=0}^{\infty} k^2 x^k \{ \cos k(s-t) - \cos k(s+t) \} \\ &= \frac{1}{2} \frac{\partial}{\partial s} (1-x) \sum_{k=0}^{\infty} k x^k \{ \sin k(s-t) - \sin k(s+t) \} \\ &= \frac{\pi}{4} \frac{\partial}{\partial s} \{ P_1(x, s-t) - P_1(x, s+t) \}, \end{aligned}$$

we have

$$J_1(x) = \frac{\pi}{4T} \int_0^\pi \int_0^\pi \left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 P_2(x, s+t) ds dt \\ - \frac{\pi}{4T} \int_0^\pi \int_0^\pi \left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 P_2(x, s-t) ds dt.$$

Here we write $P_2(x, s) = \partial_s P_1(x, s)$. We write $K_1(x)$ for the first term and $K_2(x)$ for the second term.

Proposition 5.7. *The following (5.2) - (5.4) hold.*

$$\lim_{x \rightarrow 1-0} \int_0^\pi v |P_2(x, v)| dv < \infty. \quad (5.2)$$

$$\lim_{x \rightarrow 1-0} \int_0^\pi v P_2(x, v) dv = -\frac{2}{\pi}. \quad (5.3)$$

$$\lim_{x \rightarrow 1-0} \int_\delta^\pi v |P_2(x, v)| dv = 0 \text{ for all } \delta > 0. \quad (5.4)$$

As in the proof of Lemma 5.6, for all $\phi \in C[0, \pi]$ it follows that

$$\lim_{x \rightarrow 1-0} \int_0^\pi v P_2(x, v) \phi(v) dv = -\frac{2}{\pi} \phi(0).$$

Proof. Proof of (5.2): Let $0 \leq \theta_x \leq \pi$ be such as $P_2(x, \theta_x) = 0$. Since

$$P_2(s, x) = \frac{2}{\pi} x(1-x)(1-x^2) \left\{ \frac{\cos s}{(1-2x \cos s + x^2)^2} - \frac{4x \sin^2 s}{(1-2x \cos s + x^2)^3} \right\},$$

we have

$$\cos \theta_x = \frac{\sqrt{(1+x^2)^2 + 32x^2} - (1+x^2)}{4x}.$$

For fixed $0 \leq x < 1$, hence by $P_2(x, v) \geq 0$ if and only if $0 \leq v \leq \theta_x$, we have

$$\int_0^\pi v |P_2(x, v)| dv = \int_0^{\theta_x} v P_2(x, v) dv - \int_{\theta_x}^\pi v P_2(x, v) dv.$$

Here the primitive integral of $v P_2(x, v)$ is calculated by

$$\int v P_2(x, v) dv = \int v P_1'(x, v) dv = v P_1(x, v) - \int P_1(x, v) dv \\ = v P_1(x, v) + (1-x) \int P_0'(x, v) dv = v P_1(x, v) + (1-x) P_0(x, v).$$

So we have

$$\int_0^\pi \int_0^\pi v |P_2(x, v)| dv = 2\theta_x P_1(x, \theta_x) + 2(1-x) P_0(x, \theta_x) \\ - (1-x) P_0(x, 0) - \pi P_1(x, \pi) - (1-x) P_0(x, \pi)$$

For the first two terms, by elementary computation it is shown that

$$\begin{aligned}\theta_x P_1(x, \theta_x) &= \frac{1}{4\pi} \frac{x}{1+x} \frac{\theta_x}{\sin \theta_x} \frac{6(1+x^2)\sqrt{(1+x^2)^2+32x^2}+10(1+x^2)^2+32x^2}{(1+x^2)\sqrt{(1+x^2)^2+32x^2}+(1+x^2)^2+8x^2} \\ (1-x)P_0(x, \theta_x) &= \frac{1}{\pi} \frac{3(1+x^2)+\sqrt{(1+x^2)^2+32x^2}}{4(1+x)}.\end{aligned}$$

Since $\theta_x \rightarrow +0$ as $x \rightarrow 1-0$, we have $2\theta_x P_1(x, \theta_x) \rightarrow \frac{3}{2\pi}$ and $2(1-x)P_0(x, \theta_x) \rightarrow \frac{3}{\pi}$ as $x \rightarrow 1-0$. $(1-x)P_0(x, 0) = \frac{1+x}{\pi} \rightarrow \frac{2}{\pi}$ and the last two terms are easily shown to converge to 0. Combining them all, we have

$$\lim_{x \rightarrow 1-0} \int_0^\pi v |P_2(x, v)| dv = \frac{3}{2\pi} + \frac{3}{\pi} - \frac{2}{\pi} = \frac{5}{2\pi} < \infty.$$

Proof of (5.3): Because of

$$\int_0^\pi v P_2(x, v) dv = \pi P_1(x, \pi) + (1-x)P_0(x, \pi) - (1-x)P_0(x, 0),$$

we have

$$\lim_{x \rightarrow 1-0} \int_0^\pi v P_2(x, v) dv = - \lim_{x \rightarrow 1-0} (1-x)P_0(x, 0) = -\frac{2}{\pi}.$$

Proof of (5.4): Since $\theta_x \rightarrow 0$ as $x \rightarrow 1-0$, we are able to take $0 \leq x < 1$ to be $\theta_x < \delta$ for all $\delta > 0$. Then because $P_2(x, v) \leq 0$ when $\theta_x \leq v \leq \pi$, we obtain

$$\begin{aligned}\int_\delta^\pi v |P_2(x, v)| dv &= - \int_\delta^\pi v P_2(x, v) dv \\ &= \pi P_1(x, \pi) + (1-x)P_0(x, \pi) - \delta P_1(x, \delta) - (1-x)P_0(x, \delta).\end{aligned}$$

It is easy to check that the last four terms converge to 0 as $x \rightarrow 1-0$, which proves the lemma. \square

Corollary 5.8.

$$\lim_{x \rightarrow 1-0} K_1(x) = 0.$$

Proof. Let $A_\delta^1 = \{0 \leq s \leq \pi, 0 \leq t \leq \pi, 0 \leq s+t < \delta\}$, $A_\delta^2 = \{0 \leq s \leq \pi, 0 \leq t \leq \pi, 2\pi - \delta < s+t \leq 2\pi\}$, and $D_\delta = \{0 \leq s \leq \pi, 0 \leq t \leq \pi, \delta \leq s+t \leq 2\pi - \delta\}$. The integral domain of $K_1(x)$ is divided into A_δ^1 , A_δ^2 , and D_δ . We call the integral over A_δ^1 , A_δ^2 , and D_δ as $L_1(x)$, $L_2(x)$ and $L_3(x)$ respectively. We first compute $L_1(x)$.

$$|L_1(x)| \leq \sup_{(s,t) \in A_\delta^1} \left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 \iint_{A_\delta^1} |P_2(x, s+t)| ds dt.$$

Since $(s, t) \in A_\delta^1$ implies $0 \leq s \leq \delta$, $0 \leq t \leq \delta$ and

$$\left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right| \leq \left| \phi\left(\frac{T}{\pi}s\right) - \phi(0) \right| + \left| \phi(0) - \phi\left(\frac{T}{\pi}t\right) \right|,$$

we have

$$\sup_{(s,t) \in A_\delta^1} \left| \phi\left(\frac{T}{\pi}s\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 \leq 4 \sup_{0 \leq u \leq \pi T^{-1}\delta} |\phi(u) - \phi(0)|^2.$$

On the other hand, we have

$$\begin{aligned} \iint_{A_\delta^1} |P_2(x, s+t)| ds dt &\leq \int_0^\pi \int_0^\pi |P_2(x, s+t)| ds dt \\ &= \int_0^\pi \int_0^v |P_2(x, v)| ds dv + \int_\pi^{2\pi} \int_{v-\pi}^\pi |P_2(x, v)| ds dv \quad (\text{Set } v = s+t) \\ &= \int_0^\pi v |P_2(x, v)| dv + \int_\pi^{2\pi} (2\pi - v) |P_2(x, v)| dv = 2 \int_0^\pi v |P_2(x, v)| dv. \end{aligned}$$

The last equation follows from $P_2(x, v) = P_2(x, 2\pi - v)$. Because of (5.2), the last term converges to $5/\pi$ as $x \rightarrow 1 - 0$. Thus we have

$$|L_1(x)| \leq \frac{20}{\pi} \sup_{0 \leq u \leq \pi T^{-1}\delta} |\phi(u) - \phi(0)|^2.$$

In the same manner we can show that

$$|L_2(x)| \leq \frac{20}{\pi} \sup_{0 \leq u \leq \pi T^{-1}\delta} |\phi(T - u) - \phi(T)|^2.$$

We proceed to compute $L_3(x)$. We have

$$\begin{aligned} L_3(x) &\leq 4 \sup_{0 \leq u \leq T} |\phi(u)|^2 \iint_{D_\delta} |P_2(x, s+t)| ds dt \\ &= 4 \sup_{0 \leq u \leq T} |\phi(u)|^2 \left(\int_\delta^\pi \int_0^v |P_2(x, v)| ds dv + \int_\pi^{2\pi-\delta} \int_{v-\pi}^\pi |P_2(x, v)| ds dv \right) \\ &= 8 \sup_{0 \leq u \leq T} |\phi(u)|^2 \int_\delta^\pi v |P_2(x, v)| dv. \end{aligned}$$

Because of (5.4), the last term of the equation converges to 0 as $x \rightarrow 1 - 0$ for all $\delta > 0$. Taken together, we have

$$\lim_{x \rightarrow 1-0} |K_1(x)| \leq \frac{20}{\pi} \left(\sup_{0 \leq u \leq \pi T^{-1}\delta} |\phi(u) - \phi(0)|^2 + \sup_{0 \leq u \leq \pi T^{-1}\delta} |\phi(T - u) - \phi(T)|^2 \right).$$

Since ϕ is uniformly continuous on $[0, T]$, the left hand side goes to 0 as $\delta \rightarrow +0$. Thus we obtain

$$\lim_{x \rightarrow 1-0} K_1(x) = 0. \quad \square$$

We now turn to deal with $K_2(x)$. By letting $v = s - t$, the domain of integration $[0, \pi] \times [0, \pi]$ is transformed to $\{v \leq t \leq \pi - v, 0 \leq v \leq \pi\} \cup \{-v \leq t \leq \pi + v, -\pi \leq v \leq 0\}$. Since the integrand is symmetric with respect to $v = 0$, we have

$$\lim_{x \rightarrow 1-0} K_1(x) \leq \frac{\pi}{2T} \int_0^\pi \int_v^{\pi-v} \left| \phi\left(\frac{T}{\pi}(t+v)\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 P_2(x, v) dt dv. \quad (5.5)$$

Set

$$I(v) = -\frac{\pi}{2T} \int_0^{\pi-v} \frac{\left| \phi\left(\frac{T}{\pi}(t+v)\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2}{|v|} dt.$$

Then $K_1(x)$ is written by

$$K_2(x) = \int_0^\pi v P_2(x, v) I(v) dv.$$

Proposition 5.7 shows that $\lim_{x \rightarrow 1-0} K_2(x) = -\frac{2}{\pi} \lim_{v \rightarrow 0} I(v)$ if the limit of the right hand side exists.

Let N be a integer. By substituting $\pi/2^N$ for v , we have

$$\begin{aligned} I\left(\frac{\pi}{2^N}\right) &= -\frac{\pi}{2T} \int_0^{\pi-\frac{\pi}{2^N}} \left| \phi\left(\frac{T}{\pi}t + \frac{T}{2^N}\right) - \phi\left(\frac{T}{\pi}t\right) \right|^2 \frac{2^N}{\pi} dt \\ &= -\frac{\pi^2}{2T^2} \int_0^{T-\frac{T}{2^N}} \left| \phi\left(t + \frac{T}{2^N}\right) - \phi(t) \right|^2 \frac{2^N}{\pi} dt. \end{aligned}$$

We will consider the limit of $I(\pi/2^N)$ as $N \rightarrow \infty$.

Let $M(> N)$ be an integer. We write $I_{M,N}$ for a Riemann sum approximations to the integral $I\left(\frac{\pi}{2^N}\right)$ such that the interval $[0, T - \frac{T}{2^N}]$ is divided into 2^{M-N} subintervals, all with the same length $\frac{T}{2^M}$, i.e.

$$I_{M,N} = -\frac{\pi}{2T} \sum_{k=0}^{(2^N-1)2^{M-N}} \left| \phi\left(\frac{kT}{2^M} + \frac{T}{2^N}\right) - \phi\left(\frac{kT}{2^M}\right) \right|^2 \frac{2^N}{2^M}.$$

By letting $k = (p-1)2^{M-N} + r$, $p = 1, 2, \dots, 2^N - 1$, $r = 1, 2, \dots, 2^{M-N}$, we change the order of sum of $I_{M,N}$.

$$I_{M,N} = -\frac{\pi}{2T} \left\{ \sum_{r=1}^{2^{M-N}} \sum_{p=1}^{2^N-1} \left| \phi\left(\frac{pT}{2^N} + \frac{rT}{2^M}\right) - \phi\left(\frac{(p-1)T}{2^N} + \frac{rT}{2^M}\right) \right|^2 + \left| \phi(0) - \phi\left(\frac{T}{2^M}\right) \right|^2 \right\} \frac{2^N}{2^M}$$

Here we focus on the term

$$J_r = \sum_{p=1}^{2^N-1} \left| \phi\left(\frac{pT}{2^N} + \frac{rT}{2^M}\right) - \phi\left(\frac{(p-1)T}{2^N} + \frac{rT}{2^M}\right) \right|^2.$$

By definition we have

$$\begin{aligned}
J_r &\leq \sup_{|\Delta| \leq \frac{T}{2^N}} Q_T(\phi, \Delta) \\
J_r &\geq \inf_{|\Delta| \leq \frac{T}{2^N}} Q_T(\phi, \Delta) - \left\{ \left| \phi(0) - \phi\left(\frac{rT}{2^M}\right) \right|^2 + \left| \phi(T) - \phi\left(T - \frac{(2^{M-N} - r)T}{2^M}\right) \right|^2 \right\} \\
&\geq \inf_{|\Delta| \leq \frac{T}{2^N}} Q_T(\phi, \Delta) - \sup_{0 \leq v \leq \frac{T}{2^N}} \{ |\phi(0) - \phi(v)|^2 - |\phi(T) - \phi(T-v)|^2 \}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\inf_{|\Delta| \leq \frac{T}{2^N}} Q_T(\phi, \Delta) - \sup_{0 \leq v \leq \frac{T}{2^N}} \{ |\phi(0) - \phi(v)|^2 - |\phi(T) - \phi(T-v)|^2 \} \\
&\leq -\frac{2T}{\pi} I\left(\frac{\pi}{2^N}\right) \leq \sup_{|\Delta| \leq \frac{T}{2^N}} Q_T(\phi, \Delta)
\end{aligned} \tag{5.6}$$

as $M \rightarrow \infty$. By letting $N \rightarrow \infty$, we can conclude that $\lim_{v \rightarrow +0} I(v) = -\frac{\pi \langle \phi \rangle_T}{2T}$ if the quadratic variation of ϕ over $[0, T]$ exists. Thus we obtain

$$\lim_{x \rightarrow 1-0} K_2(x) = \frac{\langle \phi \rangle_T}{T},$$

which proves the theorem 5.3. □

To apply theorem 5.3 to the Lévy Laplacian for square roots of measures, we need to check $\frac{1}{n} \sum_{k=0}^n \langle \phi, e_k \rangle^2$ converges in the norm topology of $L^2(D'[0, T])$.

Lemma 5.9. *Let f be second order Fréchet differentiable in the direction of H_a , and $|\hat{f}|^2(D'[0, T] \setminus C_0[0, T]) = 0$. Then*

$$\int_{C_0[0, T]} |\phi(s)|^4 |\hat{f}|^2(d\phi) < \infty$$

for $0 \leq s \leq T$.

Proof. Let $\eta_s^\epsilon \in H_0^1[0, T]$ ($0 \leq s \leq T$, $\epsilon > 0$) to be the function

$$\eta_s^\epsilon(u) = \begin{cases} u & (0 \leq u \leq s - \epsilon) \\ -\frac{1}{2\epsilon}(u - s)^2 + s - \frac{\epsilon}{2} & (s - \epsilon \leq u \leq s) \\ s - \frac{\epsilon}{2} & (s \leq u \leq T). \end{cases}$$

Since η_s^ϵ is second order differentiable, we have

$$\langle \phi, \eta_s^\epsilon \rangle = [\phi(u)\dot{\eta}_s^\epsilon(u)]_0^T - \int_0^T \phi(u)\ddot{\eta}_s^\epsilon(u)du = \frac{1}{\epsilon} \int_{s-\epsilon}^\epsilon \phi(u)du$$

for $\phi \in C_0[0, T]$. Thus we obtain

$$\|\partial_{\eta_s^\epsilon}^2 f\|^2 = (2\pi)^4 \int_{C_0[0, T]} \left(\frac{1}{\epsilon} \int_{s-\epsilon}^\epsilon \phi(u)du \right)^4 |\hat{f}|^2(d\phi).$$

Because f is second order Fréchet differentiable in the direction of H_a , the left hand side converges to $d^2 f(\eta_s, \eta_s)$ as $\epsilon \rightarrow 0$. Here

$$\eta_s(u) = \begin{cases} u & (0 \leq u \leq s) \\ s & (s \leq u \leq T). \end{cases}$$

Hence by

$$\lim_{\epsilon \rightarrow +0} \frac{1}{\epsilon} \int_{s-\epsilon}^\epsilon \phi(u)du = \phi(s)$$

for $\phi \in C_0[0, T]$, we have

$$\lim_{\epsilon \rightarrow 0} (2\pi)^4 \int_{C_0[0, T]} \left(\frac{1}{\epsilon} \int_{s-\epsilon}^\epsilon \phi(u)du \right)^4 |\hat{f}|^2(d\phi) \geq (2\pi)^4 \int_{C_0[0, T]} |\phi(s)|^4 |\hat{f}|^2(d\phi),$$

which proves the lemma. \square

Theorem 5.10. *Let $f \in L^2(D'[0, T])$ be twice differentiable in any direction of $H_0^1[0, T]$, let A be the totality of $\phi \in C_0[0, T]$ such that the quadratic variation of ϕ exists and let $D(\Delta_L)$ be*

$$D(\Delta_L) = \{f \in L^2(D'[0, T]) : |\hat{f}|^2(D'[0, T] \setminus A) = 0, \int_A \langle \phi \rangle^2 |\hat{f}|^2(d\phi) < \infty\}.$$

If $f \in D(\Delta_L)$, then $\Delta_L f$ exists and it follows that

$$\mathcal{F}(\Delta_L f) = -(2\pi)^2 \frac{\langle \cdot \rangle_T}{T} \times \hat{f}.$$

Proof. It is sufficient to show that

$$\lim_{x \rightarrow 1-0} \int_A \left| K_1(x) + K_2(x) - \frac{\langle \phi \rangle_T}{T} \right|^2 |\hat{f}|^2(d\phi) = 0 \quad (5.7)$$

for $f \in D(\Delta_L)$. We first show that $\lim_{x \rightarrow 1-0} \int_A |K_1(x)|^2 |\hat{f}|^2(d\phi) = 0$. Hence by the proof of Lemma 5.8, we have

$$\begin{aligned} \lim_{x \rightarrow 1-0} \int_A |K_1(x)|^2 |\hat{f}|^2(d\phi) &\leq C \int_A \sup_{0 \leq u \leq \pi T^{-1}\delta} (|\phi(u) - \phi(0)|^4 + |\phi(T-u) - \phi(T)|^4) |\hat{f}|^2(d\phi) \\ &\quad + C \lim_{x \rightarrow 1-0} \int_A \sup_{0 \leq u \leq T} |\phi(u)|^4 |\hat{f}|^2(d\phi) \left(\int_\delta^\pi v |P_2(x, v)| dv \right)^2 \end{aligned}$$

for some $C > 0$ and for all $\delta > 0$. Lemma 5.9 shows that the above integrals are finite. Since $\sup_{0 \leq u \leq \pi T^{-1}\delta} (|\phi(u) - \phi(0)|^4 + |\phi(T-u) - \phi(T)|)$ is monotonically decreasing to 0 as $\delta \rightarrow +0$, the first term goes to 0. Because of (5.4), the second term converges to 0 as $x \rightarrow +0$.

We next show that $\lim_{x \rightarrow 1-0} \int_A \left| K_2(x) - \frac{\langle \phi \rangle_T}{T} \right|^2 |\hat{f}|^2(d\phi) = 0$. Hence by (5.6), it follows that

$$\begin{aligned} \lim_{x \rightarrow 1-0} \left| K_2(x) - \frac{\langle \phi \rangle_T}{T} \right| &\leq \lim_{v \rightarrow +0} \left| \sup_{|\Delta| \leq v} Q_T(\phi, \Delta) - \frac{\langle \phi \rangle_T}{T} \right| + \lim_{v \rightarrow +0} \left| \inf_{|\Delta| \leq v} Q_T(\phi, \Delta) - \frac{\langle \phi \rangle_T}{T} \right| \\ &\quad + \lim_{v \rightarrow +0} \sup_{0 \leq u \leq v} (|\phi(u) - \phi(0)| + |\phi(T-u) - \phi(T)|) \\ &= \lim_{v \rightarrow +0} \sup_{|\Delta| \leq v} Q_T(\phi, \Delta) - \lim_{v \rightarrow +0} \inf_{|\Delta| \leq v} Q_T(\phi, \Delta) \\ &\quad + \lim_{v \rightarrow +0} \sup_{0 \leq u \leq v} (|\phi(u) - \phi(0)| + |\phi(T-u) - \phi(T)|). \end{aligned}$$

Since the last term is monotonically decreasing to 0 as $\delta \rightarrow +0$, monotone convergence theorem shows the desired conclusion. Taken together, (5.7) is proven. \square

Our next objective is to characterize the Lévy Laplacian for square roots of measures via asymptotic spherical mean. Let S_n be the n -dimensional unit sphere and μ_n be the normalized uniform measure on S_n .

Definition 5.11. Let $f \in L^2(C_0[0, T])$ be $H_0^1[0, T]$ -shift continuous. Via Bochner integral of $\tau_{\rho h(n)} f$ over S_n , we define

$$M_\rho^n f = \int_{S_{n-1}} \tau_{\rho h(n)} f d\mu_{n-1}.$$

Here we write $h^{(n)} = \sum_{k=1}^n h_k e_k$. If $M_\rho^n f$ converges some square root of a measure as $n \rightarrow \infty$ in the norm topology of $L^2(D'[0, T])$, this is called the asymptotic spherical mean of f over the sphere of radius ρ and it is written by $M_\rho f$.

Proposition 5.12. Let $f \in L^2(C_0[0, T])$ be $H_0^1[0, T]$ -shift continuous and $|\hat{f}|^2(D'[0, T] \setminus A) = 0$. Then $M_\rho^n f$ converges to the spherical mean $M_\rho f$ in the norm topology of $L^2(D'[0, T])$ as $n \rightarrow \infty$ and it follows that

$$\mathcal{F}(M_\rho f) = \left(e^{-2\pi^2 \rho^2 T^{-1} \langle \cdot \rangle_T} \hat{f} \right).$$

Proof. Let $f \in L^2(C_0[0, T])$ be $H_0^1[0, T]$ -shift continuous and $|\hat{f}|^2(D'[0, T] \setminus A) = 0$. Then we have

$$\begin{aligned} \mathcal{F}(M_\rho^n f) &= \mathcal{F} \left(\int_{S_{n-1}} \tau_{\rho h(n)} f d\mu_{n-1} \right) = \int_{S_{n-1}} \mathcal{F}(\tau_{\rho h(n)} f) d\mu_{n-1} \\ &= \int_A \int_{S_{n-1}} e^{2\pi\sqrt{-1}\rho \langle h^{(n)}, \phi \rangle} \mu_{n-1}(dh^{(n)}) |\hat{f}|^2(d\phi) \end{aligned}$$

Let w_n be the volume of the surface area of S_n and $r_n^2 = \sum_{k=1}^n \langle \phi, e_k \rangle^2$. Choosing some adequate unitary transform on S_n , we have

$$\begin{aligned} \int_{S_{n-1}} e^{2\pi\sqrt{-1}\rho\langle h^{(n)}, \phi \rangle} \mu(dh^{(n)}) &= \frac{1}{w_{n-1}} \int_{S_{n-1}} e^{2\pi\sqrt{-1}\rho r_n h_n} dh_1 \dots dh_n \\ &= \frac{w_{n-2}}{w_{n-1}} \int_{-1}^1 (1-x^2)^{\frac{n}{2}-1} e^{2\pi\sqrt{-1}\rho r_n x} dx = 2 \frac{w_{n-2}}{w_{n-1}} \int_0^1 (1-x^2)^{\frac{n}{2}-1} \cos(2\pi\rho r_n x) dx. \end{aligned}$$

Let $I_n(\phi)$ be the right hand side of the above equation. Then we have

$$\begin{aligned} I_n(\phi) &= 2 \frac{w_{n-2}}{w_{n-1}} \sum_{k=0}^{\infty} \frac{(-4\pi^2 \rho^2 r_n^2)^k}{(2k)!} \int_0^1 x^{2k} (1-x^2)^{\frac{n}{2}-1} dx \\ &= \frac{w_{n-2}}{w_{n-1}} \sum_{k=0}^{\infty} \frac{(-4\pi^2 \rho^2 r_n^2)^k}{(2k)!} \int_0^1 t^{k-\frac{1}{2}} (1-t)^{\frac{n}{2}-1} dt \quad (\text{Set } t = x^2) \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-4\pi^2 \rho^2 r_n^2)^k}{(2k)!} \frac{\Gamma(k + \frac{1}{2}) \Gamma^2(\frac{n}{2})}{\Gamma(k + \frac{n+1}{2}) \Gamma(\frac{n-1}{2})} \\ &= \sum_{k=0}^{\infty} \frac{(-\pi^2 \rho^2 r_n^2)^k}{k!} \frac{\Gamma^2(\frac{n}{2})}{(k + \frac{n-1}{2}) (k + \frac{n-3}{2}) \dots \frac{n+1}{2} \Gamma(\frac{n+1}{2}) \Gamma(\frac{n-1}{2})} \\ &= \frac{\Gamma^2(\frac{n}{2})}{\Gamma(\frac{n+1}{2}) \Gamma(\frac{n-1}{2})} \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(-\frac{2\pi^2 \rho^2 r_n^2}{n} \right)^k. \end{aligned}$$

Here we write

$$b_0 = 1, \quad b_k = \prod_{j=1}^k \left(1 + \frac{2j-1}{n} \right)^{-1} \quad (k \geq 1).$$

Since by Wallis' formula, it follows that

$$\lim_{n \rightarrow \infty} \frac{\Gamma^2(\frac{n}{2})}{\Gamma(\frac{n+1}{2}) \Gamma(\frac{n-1}{2})} = 1.$$

Because of

$$b_k - b_{k+1} = b_k \frac{2k-1}{n} \left(1 + \frac{2k-1}{n} \right)^{-1} \leq \frac{2k-1}{n},$$

we have $1 - b_k \leq k^2/n$ ($k \geq 0$). So letting $x_n = -\frac{2\pi^2 \rho^2 r_n^2}{n}$ we have

$$\left| \sum_{k=0}^{\infty} \frac{1-b_k}{k!} x_n^k \right| \leq \frac{1}{n} \sum_{k=0}^{\infty} \frac{k}{(k-1)!} |x_n|^k = \frac{1}{n} (|x_n| + 1) e^{|x_n|}.$$

By assumption the right hand side converges to 0 for $\phi \in A$. Thus it follows that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{b_k}{k!} x_n^k = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{k!} x_n^k = e^{-2\pi^2 \rho^2 T^{-1} \langle \phi \rangle_T} \quad (5.8)$$

for $\phi \in A$. In addition, because of

$$|I_n(\phi)| = \left| 2 \frac{w_{n-2}}{w_{n-1}} \int_0^1 (1-x^2)^{\frac{n}{2}-1} \cos(2\pi \rho r_n x) dx \right| \leq 2 \frac{w_{n-2}}{w_{n-1}} \int_0^1 (1-x^2)^{\frac{n}{2}-1} dx = 1,$$

$I_n(\phi)$ is uniformly bounded. We are now in a position to show $\{M_\rho^n f\}$ is a Cauchy sequence in the norm topology of $L^2(D'[0, T])$. Let m, n ($m > n$) be integers. Because $I_n(\phi)$ converges to $e^{-2\pi^2 \rho^2 T^{-1} \langle \phi \rangle_T}$ for all $\phi \in A$ as $n \rightarrow \infty$ and $I_n(\phi) \leq 1$, we have

$$\lim_{m, n \rightarrow \infty} \|M_\rho^n f - M_\rho^m f\|^2 = \lim_{m, n \rightarrow \infty} \int_A |I_m(\phi) - I_n(\phi)|^2 |\hat{f}|^2(d\phi) = 0,$$

which proves the proposition. \square

The relation between the spherical mean and the Lévy Laplacian is stated as follows.

Corollary 5.13. *Assume that $f \in D(\Delta_L)$. Then we have*

$$\Delta_L f = 2 \lim_{\rho \rightarrow +0} \frac{M_\rho f - f}{\rho^2}.$$

Proof. Let $f \in D(\Delta_L)$. Theorem 5.10 and Proposition 5.12 shows that

$$\left\| \Delta_L f - 2 \frac{M_\rho f - f}{\rho^2} \right\|^2 = (2\pi)^2 \int_A \left| \frac{\langle \phi \rangle_T}{T} + \frac{e^{-2\pi^2 \rho^2 T^{-1} \langle \phi \rangle_T} - 1}{2\pi^2 \rho^2} \right|^2 |\hat{f}|^2(d\phi).$$

Letting $\rho \rightarrow 0$, we have the desired conclusion. \square

References

- [1] L. Accardi *On square roots of measures* Proc. Internat. School of Physics “Enrico Fermi”, Course LX, North-Holland, (1976), 167-189
- [2] Averbuh, V.I., Smolyanov, O.G., and Fomin, S.V. *Generalized functions and differential equations in linear spaces, T. Differentiable measures* Trans. Moscow Math Soc., **24** (1971), 140-184.
- [3] S. Bochner *Harmonic analysis and the theory of probability* Univ. of California, 1955.
- [4] N. Bourbaki, *Intégration*, 2nd edition, Hermann, Paris, Chapters I-V, Chapter IX. 1965, 1969.
- [5] R. Cameron and W. T. Martin, *Transformations of Wiener Integrals under translations*, Ann. of Math. **45** (1944), 386-396.
- [6] R. Cameron and W. T. Martin, *Fourier-Wiener transformations of functionals belong to L_2 over the space C* , Duke Math. **14**(1947), 99-107

- [7] N. Dunford, and J. T. Schwartz, *Linear operators, Part I. General theory*, Wiley-Interscience, New York, 1967.
- [8] T. Hida, *Analysis of Brownian Functionals*, Carleton Mathematics Lecture Notes, Vol. 13, 1975; 2nd ed., 1978.
- [9] T. Hida, *Brownian Motion*, Springer-Verlag, 1980.
- [10] H. Kakuma, *Fourier Transform of Square Roots of Measures on Infinite Dimensional Spaces*, submitted to Infinite Dimensional Analysis, Quantum Probability and Related Topics.
- [11] S. Kakutani *On equivalence of infinite product measures*, Ann. of Math. **49** (1948), 214-224.
- [12] A. N. Kolmogorov *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, 1933.
- [13] H. H. Kuo *White Noise Distribution Theory*, CRC press Inc, 1996
- [14] H. H. Kuo *Brownian functionals and applications*, Acta Appl. Math. **1**(1983), 175-188.
- [15] R. A. Minlos *Generalized random processes and their extension to a measure*, Selected transl. in Math. Statist. and Probability. **3**(1962), 291-313.
- [16] E. Nelson *Quantum fields and Markof fields*, in Proceedings of the 1971, Summer Conference.
- [17] N. Obata *Analysis of the Lévy Laplacian*, Soochow J. Math. **14** (1988), 105-109
- [18] K. R. Parthasarathy, *Probability measures on metric space*, Academic Press, New York, 1967.
- [19] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, volume 44 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.
- [20] L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford Univ. Press, London, 1973.
- [21] I. E. Segal *Tensor algebras over Hilbert spaces. I* Trans. Amer. Math. Soc. **81**, 106 (1956).
- [22] A. V. Skorohod, *Integration in Hilbert spaces*, Springer, Berlin-New York, 1974.
- [23] E. G. F. Thomas *Projective limits of complex measures and martingale convergence* Probab. Theory Relat. Fields. **119**(2001), 579-588
- [24] V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. (N.S.) **55** (97) (1961), 35-100. (AMS translation, Ser. 2, Math. USSR-Sb., **48**, 161-228.)
- [25] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, 1940
- [26] N. Wiener, *Differential space*, J. Math. Phys., **58** (1923), 131-174.

- [27] D. X. Xia, *Measure and integration theory on infinite-dimensional spaces. Abstract harmonic analysis*, Pure and Applied Mathematics, Vol. 48. Academic Press, New York-London, 1972.
- [28] Y. Yamasaki, *Measures on infinite-dimensional spaces*, Series in Pure Mathematics, 5. World Scientific Publishing Co., Singapore, 1985.
- [29] K. Yosida, *Functional Analysis*, Springer, 1965.